

On the theorem of the heart in negative K -theory

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Abstract

Schlichting conjectured that the negative K -groups of small abelian categories vanish and proved this for noetherian abelian categories and for all abelian categories in degree -1 . The main results of this paper are that $K_{-1}(E)$ vanishes when E is a small stable ∞ -category with a bounded t -structure and that $K_{-n}(E)$ vanishes for all $n \geq 1$ when additionally the heart of E is noetherian. It follows that Barwick's theorem of the heart holds for nonconnective K -theory spectra when the heart is noetherian. We give several applications. On the one hand, we prove the vanishing results for the negative K -groups of a large class of dg algebras. On the other, we prove that various localization sequences à la Blumberg-Mandell and Barwick-Lawson hold in nonconnective K -theory. This leads to the first computations in negative K -theory of nonconnective ring spectra arising in chromatic homotopy theory. For example, $K_{-n}(\mathrm{KU})$ vanishes for all $n \geq 1$.

Key Words. Negative K -theory, t -structures, abelian categories, K -theory of dg algebras and ring spectra.

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1 Introduction

Consider the following conjectures about negative and nonconnective K -theory.

Conjecture A. *If A is a small abelian category, then $K_{-n}(A) = 0$ for $n \geq 1$.*

Conjecture B. *If E is a small stable ∞ -category with a bounded t -structure, then $K_{-n}(E) = 0$ for $n \geq 1$.*

Conjecture C. *If E is a small stable ∞ -category with a bounded t -structure, then the natural map $K(E^\heartsuit) \rightarrow K(E)$ is an equivalence of nonconnective K -theory spectra.*

Note that the conjectures and results of this paper apply equally well to pretriangulated dg categories with bounded t -structures.

The first conjecture is due to Schlichting. It is a special case of the second conjecture by setting $E = \mathcal{D}^b(A)$ and we will therefore refer to the second as the generalized Schlichting conjecture. Barwick proved Conjecture C in [Bar15] for connective K -theory: $K^{\text{cn}}(E^\heartsuit) \simeq K^{\text{cn}}(E)$, which generalizes the Gillet-Waldhausen theorem [TT90, Theorem 1.11.7] in the case that $E = \mathcal{D}^b(A)$. So, the open part of that conjecture may be rephrased as saying that $K_{-n}(E^\heartsuit) \rightarrow K_{-n}(E)$ is an isomorphism for all $n \geq 1$. Of course, this would follow from Conjecture B together with Conjecture A. In fact, Conjecture B holds if and only if Conjectures A and C hold.

Conjectures B and C are new. While the connective analogue of Conjecture C was expected for connective K -theory by Neeman and proved by Barwick, no precedents appear for negative K -groups.

In this paper, we prove Conjecture B for $n = 1$ in general and for all $n \geq 1$ when E^\heartsuit is noetherian. Together with results of Schlichting [Sch06], which verify Conjecture A in the same cases, this proves Conjecture C when E^\heartsuit is noetherian, and it proves that $K_{-1}(E^\heartsuit) \rightarrow K_{-1}(E)$ is always an isomorphism (both sides are zero).

Theorem 1.1. *If E is a small stable ∞ -category with a bounded t -structure, then $K_{-1}(E) = 0$.*

Theorem 1.2. *If E is a small stable ∞ -category equipped with a bounded t -structure such that E^\heartsuit is noetherian, then $K_{-n}(E) = 0$ for $n \geq 1$.*

Theorem 1.3 (Nonconnective theorem of the heart). *If E is a small stable ∞ -category with a bounded t -structure such that E^\heartsuit is noetherian, then the natural map*

$$K(E^\heartsuit) \xrightarrow{\simeq} K(E)$$

is an equivalence.

In fact, our proof of the nonconnective theorem of the heart in negative degrees is independent of Barwick's proof in nonnegative degrees, and uses the fact that the negative K -groups on both sides are zero. Thus the nonconnective theorem is really the result of gluing Barwick's theorem of the heart with our Theorem 1.2.

The proof of Theorem 1.2 is based on induction, with the base case provided by Theorem 1.1. This is the analogue of a result of Schlichting [Sch06, Theorem 6], which proved that $K_{-1}(A) = K_{-1}(\mathcal{D}^b(A)) = 0$ when A is an arbitrary small abelian category. The fact that no additional hypotheses are necessary lends credence to the three conjectures. The proof of Schlichting's result that $K_{-1}(A) = 0$ for general abelian categories A is not hard, but the proof of Theorem 1.1 is more difficult as it is necessary to find an excisive square playing the same role for E that the square

$$\begin{array}{ccc} \mathcal{D}^b(A) & \longrightarrow & \mathcal{D}^+(A) \\ \downarrow & & \downarrow \\ \mathcal{D}^-(A) & \longrightarrow & \mathcal{D}(A) \end{array}$$

plays for $\mathcal{D}^b(A)$.

In the inductive step, we use stable ∞ -categories of endomorphisms and automorphisms of E . We construct an exact sequence

$$\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega \rightarrow \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega \rightarrow \mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$$

of small idempotent complete stable ∞ -categories, where $\mathcal{C} = \text{Ind}(E)$ is the ind-completion of E , the superscript ω denotes the subcategory of compact objects, and $\mathcal{D}(\mathbb{A}^1, \mathcal{C}) \simeq \text{Mod}_{\mathbb{S}[s]} \otimes \mathcal{C}$, and similarly for $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$. The subscript $\{0\}$ denotes the full subcategory $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C}) \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})$ of objects killed by inverting the endomorphism s . Note that the ∞ -category $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ differs from the ∞ -category used in [BGT16] to define the *K-theory of endomorphisms*. Indeed, the *K-theory of endomorphisms of E* takes as input the ∞ -category of endomorphisms of objects of E . But, these need not be compact in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$. Conversely, the underlying object of a compact object of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ need not be compact in \mathcal{C} .

The technical input for the inductive step, proven in Corollary 3.17, is that if E is a small stable ∞ -category with a bounded t -structure such that E^\heartsuit is noetherian, then the same is true for $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$. This allows an inductive argument because $K(E)$ is a summand of $K(\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega)$ and this summand maps trivially to $K(\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega)$.

Theorem 1.2 can be extended to the case where E^\heartsuit is merely stably coherent; we do so in Section 3.5. We discuss in Sections 3.4 and 3.6 a counterexample to our approach when E^\heartsuit is not noetherian as well as several possible approaches for circumventing this problem. We hope that these more speculative sections will serve to pique the interest of readers thinking about related problems.

Amusingly, there are examples of stable ∞ -categories E with two different t -structures, one having a noetherian heart and the other having a non-noetherian heart. The standard example, due to Thomas and written down in [AP06], is $\mathcal{D}^b(\mathbb{P}^1)$, and was pointed out to us by Calabrese. This provides further evidence for Conjecture B.

Our major application of the theorems above is to several vanishing results for the negative *K-theory* of nonconnective dg algebras and ring spectra. These are the cohomological dg algebras A with $H^i(A) \neq 0$ for some $i > 0$, or the ring spectra A with $\pi_i A \neq 0$ for some $i < 0$. Unlike in the connective case, where trace methods can be used to compute nonnegative *K-groups* and results of [BGT13] can be used to compute negative *K-groups*, no previous general methods could be used to compute *K-groups* of nonconnective rings.

Our first uses a result of Keller and Nicolás [KN13] to construct t -structures for perfect complexes over a large class of dg algebras.

Corollary 1.4. *Let k be a commutative ring, and let A be a cohomological dg k -algebra such that $H^0(A)$ is semisimple, $H^i(A)$ is a finitely generated right $H^0(A)$ -module for all i , and $H^i(A) = 0$ for $i < 0$. Then, $K_{-n}(A) = 0$ for $n \geq 1$.*

Another important application, and the motivation for this project, is the following theorem, which extends work of Blumberg and Mandell.

Corollary 1.5. *There is a fiber sequence of nonconnective K -theory spectra*

$$K(\mathbb{Z}) \rightarrow K(\mathrm{ku}) \rightarrow K(\mathrm{KU}).$$

The connective version of this theorem was proved in [BM08]. Theorem 1.3 allows us to give the first complete computations of the negative K -theory of the kinds of nonconnective ring spectra that arise in chromatic homotopy theory. We give several examples of such computations below.

Corollary 1.6. *For $n \geq 1$, $K_{-n}(\mathrm{KU}) = 0$.*

Proof. Indeed, the fact that the negative K -groups of \mathbb{Z} vanish is classical, but also follows from the main theorem of Schlichting’s paper [Sch06]. Since ku is connective, $K_{-n}(\mathrm{ku}) \cong K_{-n}(\pi_0 \mathrm{ku}) = K_{-n}(\mathbb{Z})$ for $n \geq 0$ by Blumberg-Gepner-Tabuada [BGT13, Theorem 9.53]. Now, use Corollary 1.5. \square

This result is new and does not follow from Schlichting’s original result on K -theory of derived categories of noetherian abelian categories. The problem is that what appears naturally as the fiber of $K(\mathrm{ku}) \rightarrow K(\mathrm{KU})$ is the K -theory of $\mathrm{Mod}_{\mathrm{ku}}^{\omega, \beta\text{-nil}}$, the ∞ -category of compact β -nilpotent ku -module spectra. This small idempotent complete stable ∞ -category inherits a bounded t -structure from the (non-bounded) Postnikov t -structure on $\mathrm{Mod}_{\mathrm{ku}}^{\omega}$, and the heart is the category of finitely generated \mathbb{Z} -modules. Blumberg and Mandell showed (in a precursor to Barwick’s theorem of the heart) that this implies that $K^{\mathrm{cn}}(\mathbb{Z}) \simeq K^{\mathrm{cn}}(\mathrm{Mod}_{\mathrm{ku}}^{\omega, \beta\text{-nil}})$. However, without the nonconnective theorem of the heart, it was not clear whether or not to expect Corollary 1.6.

We have the following much more general vanishing result. Say that a ring spectrum is **even periodic** if $\pi_* R \cong \pi_0 R[u^{\pm 1}]$ where u is an element of even non-zero degree. Complex topological K -theory KU is the standard example. Note that our notion of even periodic is somewhat more general than the usual one, where the unit u is required to live in degree ± 2 .

Corollary 1.7. *Let R be an even periodic ring spectrum such that $\pi_0 R$ is a regular noetherian commutative ring. Then,*

$$K_{-n}(R) = 0$$

for $n \geq 1$.

In fact, Theorem 1.2 allows us to extend all of the results of the recent preprint [BL14] of Barwick and Lawson to nonconnective K -theory. The most striking examples in [BL14] are results for real K -theory KO and compactified topological modular forms Tmf . Giving results about negative K -theory for these ring spectra is interesting as KO is periodic, but not even, while Tmf is neither periodic nor connective.

Corollary 1.8. *There are fiber sequences of nonconnective K -theory spectra*

$$K(\mathbb{Z}) \rightarrow K(\mathrm{ko}) \rightarrow K(\mathrm{KO})$$

and

$$K(\mathbb{Z}) \rightarrow K(\mathrm{tmf}) \rightarrow K(\mathrm{Tmf}).$$

Corollary 1.9. *For $n \geq 1$, the negative K -groups $K_{-n}(\mathrm{KO})$ and $K_{-n}(\mathrm{Tmf})$ vanish.*

The methods in this paper can be used to prove vanishing results in negative K -theory for another large class of nonconnective ring spectra: cochain algebras on compact spaces¹.

¹By a compact space, we mean a compact object of Spc , the ∞ -category of spaces. These are precisely the spaces which are weak homotopy equivalent to retracts of finite CW complexes and are also called **finitely dominated spaces**.

Corollary 1.10. *Let X be a compact space and R a regular noetherian (discrete) commutative ring. There is an equivalence $\bigoplus_{x \in \pi_0 X} K(R) \simeq K(C^*(X, R))$ of nonconnective K -theory spectra. In particular,*

$$K_{-n}(C^*(X, R)) = 0$$

for $n \geq 1$.

Outline. Section 2 is dedicated to background on t -structures, proving several new inheritance results about t -structures, K -theoretic excisive squares, and the proof of Theorem 1.1. Section 3 contains the proofs of Theorem 1.2 and 1.3 as well as our thoughts of how one might attempt to prove Conjecture B in general. Section 4 contains our applications to the negative K -theory of ring spectra. In Appendix A, we construct a functorial ∞ -categorical model of the stable category of a Frobenius category. This is needed to check that the definition of negative K -theory we use agrees with Schlichting's.

Notation. Throughout, unless otherwise stated, we use homological indexing for chain complexes and objects in stable ∞ -categories. The ∞ -category of small stable ∞ -categories and exact functors is written $\text{Cat}_\infty^{\text{ex}}$, while the full subcategory of small *idempotent complete* stable ∞ -categories is written $\text{Cat}_\infty^{\text{perf}}$. Given a small stable ∞ -category E , we denote by \tilde{E} or E^\sim the idempotent completion of E . If $E \subseteq F$ is a fully faithful inclusion such that E is idempotent complete in F , then F/E denotes the Verdier quotient (the cofiber in $\text{Cat}_\infty^{\text{ex}}$).

If \mathcal{C} is an ∞ -category, $\text{Map}_{\mathcal{C}}(M, N)$ is the mapping space of morphisms from M to N in \mathcal{C} . Given an idempotent complete stable ∞ -category E , $K(E)$ always denotes the *nonconnective* K -theory spectrum of E , as defined in [BGT13]. We use $K^{\text{cn}}(E)$ for the connective cover of $K(E)$, the connective K -theory spectrum of E . Finally, if R is a ring spectrum, Mod_R , Alg_R , and CAlg_R denote the ∞ -categories of R -module spectra, \mathbb{E}_1 - R -algebra spectra (if R is commutative), and \mathbb{E}_∞ - R -algebra spectra (if R is commutative), respectively (even if R is discrete). If R is discrete, we let Mod_R^\heartsuit , Alg_R^\heartsuit , and CAlg_R^\heartsuit denote the ordinary categories of discrete right R -modules, discrete associative R -algebras (if R is commutative), and discrete commutative R -algebras (if R is commutative), respectively; this notation reflects the fact that the abelian category of discrete right R -modules is equivalent to the heart of the standard t -structure on the stable ∞ -category Mod_R .

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2 t -structures

After recalling t -structures in Section 2.1, we study induced t -structures on ind-completions and localizations in Section 2.2. In some cases, our results extend results in [BBD82] beyond the six-functor formalism setting. The ability to construct a t -structure on a localization in certain circumstances will be used later in the paper when we perform the inductive step in our generalization of Schlichting's theorem.

In Section 2.3, we study excisive squares in algebraic K -theory and their connection to adjointability. We prove that $K_{-1}(E) = 0$ when E is a small stable ∞ -category with a bounded t -structure in Section 2.4.

2.1 Definitions and first properties

The notion of a t -structure appears in Beilinson-Bernstein-Deligne [BBD82, Definition 1.3.1]. However, as we will work with homological indexing, Lurie's treatment in [Lur12, Definition 1.2.1.1] is more a convenient reference. If E is a stable ∞ -category and $x \in E$, we will typically write $x[n]$ for the n -fold suspension $\Sigma^n x$ of x . If $F \subseteq E$ is a full subcategory, we will also write $F[n] \subseteq E$ for the full subcategory spanned by the objects of the form $x[n]$, where x is an object of F .

Definition 2.1. A **t -structure** on a stable ∞ -category E consists of a pair of full subcategories $E_{\geq 0} \subseteq E$ and $E_{\leq 0} \subseteq E$ satisfying the following conditions:

- (1) $E_{\geq 0}[1] \subseteq E_{\geq 0}$ and $E_{\leq 0} \subseteq E_{\leq 0}[1]$;
- (2) if $x \in E_{\geq 0}$ and $y \in E_{\leq 0}$, then $\mathrm{Hom}_E(x, y[-1]) = 0$;
- (3) every $x \in E$ fits into a cofiber sequence $\tau_{\geq 0}x \rightarrow x \rightarrow \tau_{\leq -1}x$ where $\tau_{\geq 0}x \in E_{\geq 0}$ and $\tau_{\leq -1}x \in E_{\leq 0}[-1]$.

An exact functor $E \rightarrow F$ between stable ∞ -categories equipped with t -structures is **left t -exact** (resp. **right t -exact**) if it sends $E_{\leq 0}$ to $F_{\leq 0}$ (resp. $E_{\geq 0}$ to $F_{\geq 0}$). An exact functor is t -exact if it is both left and right t -exact. We set $E_{\geq n} = E_{\geq 0}[n]$ and $E_{\leq n} = E_{\leq 0}[n]$.

- Example 2.2.** (a) If A is a small abelian category, then the bounded derived ∞ -category $\mathcal{D}^b(A)$ (see Definition 3.21) admits a canonical t -structure, where $\mathcal{D}^b(A)_{\geq n}$ consists of the complexes x such that $H_i(x) = 0$ for $i < n$, and similarly for $\mathcal{D}^b(A)_{\leq n}$.
- (b) If A is a Grothendieck abelian category, then the derived ∞ -category $\mathcal{D}(A)$ admits a t -structure with the same description as the previous example. This stable ∞ -category and its t -structure are studied in [Lur12, Section 1.3.5].
- (c) If R is a connective \mathbb{E}_1 -ring spectrum, then the stable presentable ∞ -category Mod_R of right R -module spectra admits a t -structure with $(\mathrm{Mod}_R)_{\geq 0} \simeq \mathrm{Mod}_R^{\mathrm{cn}}$, the ∞ -category of connective R -module spectra. See for example [Lur12, Proposition 1.4.3.6]. We call this the **Postnikov t -structure**.

Condition (2) implies in fact that the mapping spaces $\mathrm{Map}_E(x, y[-1])$ are contractible for $x \in E_{\geq 0}$ and $y \in E_{\leq 0}$. This is not generally the case for the mapping spectra. Indeed, if A is a Grothendieck abelian category, then $\pi_0 \mathrm{Map}_{\mathcal{D}(A)}(x[-n], y[-1]) \cong \mathrm{Ext}_A^{n-1}(x, y)$ for $x, y \in A$. (See [Lur12, Proposition 1.3.5.6].)

Lemma 2.3. *The intersection $E_{\geq 0} \cap E_{\leq 0}$ is the full subcategory of $E_{\geq 0}$ consisting of discrete objects. Moreover, the intersection is an abelian category.*

Proof. See [Lur12, Warning 1.2.1.9] for the first statement and [BBD82, Théorème 1.3.6] for the second. \square

Definition 2.4. The abelian category $E_{\geq 0} \cap E_{\leq 0}$ is called the **heart** of the t -structure $(E_{\geq 0}, E_{\leq 0})$ on E , and is denoted E^{\heartsuit} .

Example 2.5. The hearts of the t -structures in Example 2.2 are A in (a), A in (b), and $\mathrm{Mod}_{\pi_0 R}^{\heartsuit}$, the abelian category of right $\pi_0 R$ -modules, in (c).

The truncations $\tau_{\geq n}x$ and $\tau_{\leq n}x$ are functorial in the sense that the inclusions $E_{\geq n} \rightarrow E$ and $E_{\leq n} \rightarrow E$ admit right and left adjoints, respectively, by [Lur12, Corollary 1.2.1.6]. Let $\pi_n x = \tau_{\geq n} \tau_{\leq n} x[-n] \in E^\heartsuit$. This functor is homological by [BBD82, Théorème 1.3.6], meaning that there are long exact sequences

$$\cdots \rightarrow \pi_{n+1}z \rightarrow \pi_n x \rightarrow \pi_n y \rightarrow \pi_n z \rightarrow \pi_{n-1}x \rightarrow \cdots$$

in E^\heartsuit whenever $x \rightarrow y \rightarrow z$ is a cofiber sequence in E .

Definition 2.6. A t -structure $(E_{\geq 0}, E_{\leq 0})$ on a stable ∞ -category is **right separated** if

$$\bigcap_{n \in \mathbb{Z}} E_{\leq n} = 0.$$

Left separated t -structures are defined similarly. Left and right separated t -structures are called **non-degenerate** in [BBD82].

Lemma 2.7. Suppose that $A = E^\heartsuit$ is the heart of a left or right separated t -structure on a stable ∞ -category E . If $0 \rightarrow x \rightarrow y \rightarrow z \rightarrow 0$ is an exact sequence in A , then $x \rightarrow y \rightarrow z$ is a cofiber sequence in E .

Proof. Assume that the t -structure is left separated. Let w be the cofiber of $x \rightarrow y$ in E . Because $E_{\geq 0} \subseteq E$ is a left adjoint, we can identify w with the cofiber of $x \rightarrow y$ in $E_{\geq 0}$. As $E_{\geq 0} \xrightarrow{\pi_0} E^\heartsuit \simeq A$ is a left adjoint, the sequence $x \rightarrow y \rightarrow \pi_0 w \rightarrow 0$ is exact. But, it is also exact on the left by hypothesis, so that $w \simeq z$, as desired. The case of right separated t -structures follows by taking opposite categories. \square

We leave the proof of the next lemma to the reader.

Lemma 2.8. Let E and F be stable ∞ -categories with t -structures where the t -structure on E is left or right separated. If $\varphi : E \rightarrow F$ is a right (resp. left) t -exact functor, then φ induces a right (resp. left) exact functor $\pi_0 \varphi : E^\heartsuit \rightarrow F^\heartsuit$.

Recall that if A is an abelian category, then $K_0(A)$ is the Grothendieck group of A , which has generators $[x]$ for $x \in A$ and relations $[y] = [x] + [z]$ whenever x, y, z fit into an exact sequence $0 \rightarrow x \rightarrow y \rightarrow z \rightarrow 0$. Similarly, if E is a small stable ∞ -category, then $K_0(E)$ is the free abelian group on symbols $[x]$ for $x \in E$ modulo the relation $[y] = [x] + [z]$ whenever $x \rightarrow y \rightarrow z$ is a cofiber sequence in E .

It follows from Lemma 2.7 that there is a natural map $K_0(E^\heartsuit) \rightarrow K_0(E)$ when E is equipped with a left or right separated t -structure.

Definition 2.9. If E is a stable ∞ -category with a t -structure $(E_{\geq 0}, E_{\leq 0})$, we say that the t -structure is **bounded** if the inclusion

$$\bigcup_{n \rightarrow \infty} E_{\geq -n} \cap E_{\leq n} \rightarrow E$$

is an equivalence. Bounded t -structures are left and right separated.

Lemma 2.10. If E is a small stable ∞ -category equipped with a bounded t -structure, then the natural map $K_0(E^\heartsuit) \rightarrow K_0(E)$ is an isomorphism.

Proof. Using the boundedness of the t -structure, it is immediate that $K_0(E^\heartsuit) \rightarrow K_0(E)$ is surjective because every object of E is a finite iterated extension of objects in E^\heartsuit . On the other hand, by assigning to $x \in E$ the sum

$$\sum_{n \in \mathbb{Z}} (-1)^n [\pi_n x],$$

we obtain a map $K_0(E) \rightarrow K_0(E^\heartsuit)$, which splits the surjection. \square

2.2 Induced t -structures on ind-completions and localizations

We give several results about t -structures on stable ∞ -categories. Some of these, especially the equivalence of conditions (i) through (iv) in Proposition 2.19, have not, as far as we are aware, been proved before either for ∞ -categories or for triangulated categories, so we treat the subject in greater detail than is strictly needed for the rest of the paper. However, there is some overlap between this section and [Lur16, Appendix C] and [HPV16].

A t -structure $(E_{\geq 0}, E_{\leq 0})$ on a stable ∞ -category E is **bounded below** if the natural map

$$E^- = \bigcup_{n \in \mathbb{Z}} E_{\geq n} \rightarrow E$$

is an equivalence and **right complete** if the natural map

$$E \rightarrow \lim \left(\cdots \rightarrow E_{\geq m} \xrightarrow{\tau_{\geq m+1}} E_{\geq m+1} \rightarrow \cdots \right)$$

is an equivalence. **Bounded above** and **left complete** t -structures are defined similarly. A bounded below t -structure is right separated as is a right complete t -structure. Neither converse is true in general.

The following definitions were introduced in [Lur12, Section 1]. A t -structure on a stable presentable ∞ -category E is **accessible** if $E_{\geq 0}$ is presentable. A t -structure on a stable presentable ∞ -category E is **compatible with filtered colimits** if $E_{\leq 0}$ is closed under filtered colimits in E .

Example 2.11. Example 2.2(a) is bounded (above and below). It is neither left or right complete, nor is it accessible or compatible with filtered colimits, as these notions are reserved for presentable ∞ -categories. Examples 2.2(b) and (c) are right complete, accessible, and compatible with filtered colimits.

The following proposition also appears in [Lur16, Lemma C.2.4.3].

Proposition 2.12. *Suppose that E is a small stable ∞ -category with a t -structure. Then, $\mathrm{Ind}(E_{\geq 0}) \subseteq \mathrm{Ind}(E)$ determines the non-negative part of an accessible t -structure on $\mathrm{Ind}(E)$ which is compatible with filtered colimits and such that the inclusion functor $E \rightarrow \mathrm{Ind}(E)$ is t -exact. Moreover, if the t -structure on E is bounded below, then $\mathrm{Ind}(E)$ is right complete.*

Proof. The functor $\mathrm{Ind}(E_{\geq 0}) \rightarrow \mathrm{Ind}(E)$ is fully faithful by [Lur09, Proposition 5.3.5.11], and we let $\mathrm{Ind}(E)_{\geq 0}$ denote the essential image. Similarly, let $\mathrm{Ind}(E)_{\leq -1}$ denote the essential image of the fully faithful functor $\mathrm{Ind}(E_{\leq -1}) \rightarrow \mathrm{Ind}(E)$. We claim that this pair of subcategories defines a t -structure on $\mathrm{Ind}(E)$. Condition (1) of Definition 2.1 is immediate. Suppose that $x \simeq \mathrm{colim}_{i \in I} x_i$ is in $\mathrm{Ind}(E)_{\geq 0}$, where each x_i is in $E_{\geq 0}$, and let $y \simeq \mathrm{colim}_{j \in J} y_j$ be in $\mathrm{Ind}(E)_{\leq -1}$, with each $y_j \in E_{\leq -1}$. Then, by definition of the ind-completion of E ,

$$\mathrm{Map}_{\mathrm{Ind}(E)}(x, y) \simeq \lim_i \mathrm{colim}_j \mathrm{Map}_E(x_i, y_j),$$

which is contractible since each $\mathrm{Map}_E(x_i, y_j)$ is contractible. Hence, (2) holds. To verify condition (3), note that if $x \simeq \mathrm{colim}_{i \in I} x_i$ is a filtered colimit of objects $x_i \in E$, then

$$\mathrm{colim}_i \tau_{\geq 0} x_i \rightarrow x \rightarrow \mathrm{colim}_i \tau_{\leq -1} x_i$$

is a cofiber sequence since cofiber sequences commute with colimits. Hence, (3) holds.

To see that the t -structure is compatible with filtered colimits, note that $y \in \mathrm{Ind}(E)_{\leq -1}$ if and only if $\mathrm{Map}_{\mathrm{Ind}(E)}(x, y) \simeq 0$ for all $x \in \mathrm{Ind}(E)_{\geq 0} \simeq \mathrm{Ind}(E_{\geq 0})$. However, this latter condition holds if and only if $\mathrm{Map}_{\mathrm{Ind}(E)}(x, y) \simeq 0$ for all $x \in E_{\geq 0}$ since $\mathrm{Ind}(E_{\geq 0})$ is generated by $E_{\geq 0}$ under filtered colimits. Since the objects $x \in E_{\geq 0} \subseteq E$ are compact, this condition is closed under filtered colimits in y , as desired.

By construction, the functor $E \rightarrow \text{Ind}(E)$ is t -exact, and the t -structure on $\text{Ind}(E)$ is accessible as $\text{Ind}(E)_{\geq 0} \simeq \text{Ind}(E_{\geq 0})$ is presentable.

To finish the proof, we first show right separatedness. Suppose that y is an object of $\bigcap_{n \in \mathbb{Z}} \text{Ind}(E)_{\leq n}$. Since the objects of E are compact generators for $\text{Ind}(E)$, it is enough to show that the mapping spaces $\text{Map}_{\text{Ind}(E)}(x, y) \simeq 0$ for all $x \in E$. Fix $x \in E$. We have for all n a natural equivalence

$$\text{Map}_{\text{Ind}(E)}(x, y) \simeq \text{Map}_{\text{Ind}(E)}(\tau_{\leq n} x, y).$$

However, since the t -structure on E is bounded below, $\tau_{\leq n} x \simeq 0$ for n sufficiently small. Therefore, $\text{Map}_{\text{Ind}(E)}(x, y) \simeq 0$. Hence, $y \simeq 0$.

Since $\text{Ind}(E)_{\leq 0} \rightarrow \text{Ind}(E)$ is closed under finite coproducts and filtered colimits it is closed under countable coproducts. Therefore, it follows by the right complete version of [Lur12, Proposition 1.2.1.19] that $\text{Ind}(E)$ is right separated if and only if it is right complete. This completes the proof. \square

We will call the t -structure on $\text{Ind}(E)$ constructed in Proposition 2.12 the **induced** t -structure. The proof of the proposition does not extend to show that bounded above t -structures on E induce left complete t -structures on $\text{Ind}(E)$. The obstruction is that the inclusion of $\text{Ind}(E)_{\geq n}$ is a left adjoint rather than a right adjoint.

Corollary 2.13. *Let E be a small stable ∞ -category with a bounded t -structure. Then, E is idempotent complete.*

Proof. Let F be the idempotent completion of E . Equivalently, $F \simeq \text{Ind}(E)^\omega$, the full subcategory of compact objects of $\text{Ind}(E)$. We claim that the t -structure on E extends to a bounded t -structure on F . It is enough to check that the truncation functors $\tau_{\leq 0}$ and $\tau_{\geq 0}$ on $\text{Ind}(E)$ preserve compact objects. But, if $x \in F$ is a summand of $y \in E$, it follows that $\tau_{\leq 0} x$ is a summand of $\tau_{\leq 0} y$, and similarly for $\tau_{\geq 0} x$. This proves that the t -structure on $\text{Ind}(E)$ restricts to a bounded t -structure on F . The heart F^\heartsuit must be the idempotent completion of E^\heartsuit . But, since abelian categories are idempotent complete, $E^\heartsuit \rightarrow F^\heartsuit$ is an equivalence. Hence, by Lemma 2.10, $K_0(E) \rightarrow K_0(F)$ is an isomorphism. It follows from Thomason's classification of dense subcategories of triangulated categories that $E \simeq F$. See [Tho97, Theorem 2.1]. \square

In the rest of this section, we establish an important device for checking when a t -exact fully faithful functor $i : E \rightarrow F$ of small stable ∞ -categories induces a t -structure on the cofiber $G = \widetilde{F/E}$ in $\text{Cat}_\infty^{\text{perf}}$, the ∞ -category of small idempotent complete stable ∞ -categories and exact functors. Recall that G is equivalent to the idempotent completion of the Verdier localization of F by E (see [BGT13, Proposition 5.13]). We begin with a couple of easy lemmas.

Lemma 2.14. *If $i : E \rightarrow F$ is a t -exact (resp. right t -exact, resp. left t -exact) functor of stable ∞ -categories equipped with t -structures, then the induced functor $i^* : \text{Ind}(E) \rightarrow \text{Ind}(F)$ is t -exact (resp. right t -exact, resp. left t -exact) with respect to the induced t -structures on $\text{Ind}(E)$ and $\text{Ind}(F)$.*

Proof. The exactness of i^* is immediate as it preserves all small colimits and hence finite limits since $\text{Ind}(E)$ and $\text{Ind}(F)$ are stable. Because $\text{Ind}(E)_{\geq 0} \simeq \text{Ind}(E_{\geq 0})$ and $\text{Ind}(F)_{\geq 0} \simeq \text{Ind}(F_{\geq 0})$, it is immediate that $i^* : \text{Ind}(E) \rightarrow \text{Ind}(F)$ is right t -exact if i is. The same holds for left t -exactness. \square

Lemma 2.15. *Let $i : E \rightarrow F$ be a t -exact fully faithful functor of stable ∞ -categories equipped with t -structures. Then, the natural map*

$$\text{Ind}(E)^\heartsuit \rightarrow \text{Ind}(F)^\heartsuit \cap \text{Ind}(E)$$

is an exact equivalence of abelian categories.

Proof. Let $x \in \text{Ind}(F)$ be an object of the intersection. Write $x = i^*y$ for some $y \in \text{Ind}(E)$ (which is unique up to equivalence). The fact that i^* is t -exact and fully faithful implies that $\tau_{\geq 1}y \simeq 0$ and $\tau_{\leq -1}y \simeq 0$. In particular, y is contained in $\text{Ind}(E)^\heartsuit$. It follows that the map in the lemma is essentially surjective. That the map is fully faithful follows from the fact that $\text{Ind}(E) \rightarrow \text{Ind}(F)$ is fully faithful, while exactness again follows from Lemma 2.14. \square

Recall from [Lur12, Proposition 1.4.4.11] that if \mathcal{C} is a stable presentable ∞ -category and $\mathcal{C}' \subseteq \mathcal{C}$ is a full presentable subcategory closed under colimits and extensions in \mathcal{C} , then $\mathcal{C}' \simeq \mathcal{C}_{\geq 0}$ for some accessible t -structure on \mathcal{C} . We will say that the t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on \mathcal{C} is the t -structure **generated** by $\mathcal{C}' \subseteq \mathcal{C}$. This provides a way for defining many t -structures on stable presentable ∞ -categories. Note that if $\mathcal{C} \simeq \text{Ind}(E)$, where E is equipped with a t -structure $(E_{\geq 0}, E_{\leq 0})$, then the induced t -structure on $\text{Ind}(E)$ is a special case of this phenomenon: it is generated by $\text{Ind}(E_{\geq 0})$.

Definition 2.16. Let $A \subseteq B$ be an exact fully faithful functor of abelian categories. We identify A with its essential image in B . Say that A is a **weak Serre subcategory** of B if A is closed under extensions in B . We say that A is a **Serre subcategory** of B (or a **localizing subcategory** of B) if A is a weak Serre subcategory and A is additionally closed under taking subobjects and quotient objects in B .

Example 2.17. Let R be a right coherent ring. Then, the category $\text{Mod}_R^{\heartsuit, \omega}$ of finitely presented (discrete) right R -modules is an abelian subcategory of Mod_R^\heartsuit . It is always weak Serre, but it is Serre if and only if R is right noetherian.

Lemma 2.18. *Let $E \rightarrow F$ be a t -exact fully faithful functor of stable ∞ -categories equipped with left or right separated t -structures. Then, the induced map $E^\heartsuit \rightarrow F^\heartsuit$ exhibits E^\heartsuit as a weak Serre subcategory of F^\heartsuit .*

Proof. The fact that $E^\heartsuit \rightarrow F^\heartsuit$ is exact and fully faithful follows from Lemma 2.8 and the fully faithfulness of $E \rightarrow F$. To check that E^\heartsuit is closed under extensions in F^\heartsuit , consider an exact sequence $0 \rightarrow x \rightarrow y \rightarrow z \rightarrow 0$ where $x, z \in E^\heartsuit$ and $y \in F^\heartsuit$. Then, by Lemma 2.7, $x \rightarrow y \rightarrow z$ is a cofiber sequence in F . Hence, we can rewrite y as the fiber of $z \rightarrow x[1]$. Since $E \rightarrow F$ is fully faithful and preserves fibers, it follows that y is in the essential image of $E \rightarrow F$, as desired. We conclude by using Lemma 2.15. \square

The first draft of this paper contained conditions (i) through (iv) of the next proposition. Benjamin Hennion pointed out another condition, (v) below, which is shown to be equivalent to condition (iii) in [HPV16, Proposition A.5].

Proposition 2.19. *Let $i : E \rightarrow F$ be a t -exact faithful functor of stable ∞ -categories equipped with bounded t -structures, and let $j : F \rightarrow G$ be the cofiber in $\text{Cat}_\infty^{\text{perf}}$. Provide $\text{Ind}(G)$ with the accessible t -structure generated by the smallest extension-closed cocomplete subcategory of $\text{Ind}(G)$ containing the image of $F_{\geq 0}$, and equip $\text{Ind}(E)$ and $\text{Ind}(F)$ with the induced t -structures of Proposition 2.12. The following are equivalent:*

- (i) *the essential image of the embedding $i^\heartsuit : E^\heartsuit \rightarrow F^\heartsuit$ is a Serre subcategory of F^\heartsuit ;*
- (ii) *the t -structure on $\text{Ind}(G)$ restricts to a t -structure on G such that $j : F \rightarrow G$ is t -exact;*
- (iii) *the induced functor $j^* : \text{Ind}(F) \rightarrow \text{Ind}(G)$ is t -exact;*
- (iv) *the essential image of the embedding $\text{Ind}(E)^\heartsuit \rightarrow \text{Ind}(F)^\heartsuit$ is a Serre subcategory of $\text{Ind}(F)^\heartsuit$;*
- (v) *the counit map $i^*i_*x \rightarrow x$ induces a monomorphism $\pi_0(i^*i_*x) \rightarrow \pi_0(x)$ in $\text{Ind}(F)^\heartsuit$ for every object x of $\text{Ind}(F)$, where i_* is the right adjoint of $i^* : \text{Ind}(E) \rightarrow \text{Ind}(F)$.*

If these conditions hold, then the t -structure on G in (ii) is bounded.

Proof. Assume (i). Write $G' = F/E$ for the Verdier quotient of F by E . In particular, G is the idempotent completion of G' . We will construct a bounded t -structure on G' such that the functors $F \rightarrow G'$ and $G' \subseteq G \subseteq \text{Ind}(G)$ are t -exact. By Corollary 2.13, G' will be idempotent complete. This will establish (ii).

Let $L : F \rightarrow G'$ denote the quotient functor. We define $\tau_{\geq 0}Lx = L\tau_{\geq 0}x$, and similarly $\tau_{\leq 0}Lx = L\tau_{\leq 0}x$. It follows that (1) and (3) from Definition 2.1 hold trivially. Now, consider $\text{Hom}_{G'}(Lx, Ly[-1])$, where $x \in F_{\geq 0}$ and $y \in F_{\leq 0}$. Pick $f \in \text{Hom}_{G'}(Lx, Ly[-1])$. We can represent f by a zig-zag $x \leftarrow z \rightarrow y[-1]$, where the cofiber c of $x \leftarrow z$ is in E . Now, consider the following diagram

$$\begin{array}{ccccc} \tau_{\geq 0}z & \longrightarrow & z & \longrightarrow & \tau_{\leq -1}z \\ \downarrow & & \downarrow & & \downarrow \\ \tau_{\geq 0}x & \longrightarrow & x & \longrightarrow & \tau_{\leq -1}x \\ \downarrow & & \downarrow & & \downarrow \\ \tau_{\geq 0}c & \longrightarrow & c & \longrightarrow & \tau_{\leq -1}c \end{array}$$

of truncation sequences. (Warning: while the horizontal sequences are always cofiber sequences, only the central vertical sequence is a cofiber sequence in general.) The fact that $y \in F_{\leq 0}$ means that the map $z \rightarrow y[-1]$ factors through $\tau_{\leq -1}z$. Now, the fact that x is connective means that $\pi_{-n}z \in E$ for all $n \geq 1$. This is where we use the fact that E^\heartsuit is a Serre subcategory of F^\heartsuit , to ensure that the quotient $\pi_{-1}z$ of π_0c is also in E . In particular, $\tau_{\geq 0}z \rightarrow \tau_{\geq 0}x \simeq x$ has cofiber in E (though it is not in general $\tau_{\geq 0}c$). The commutative diagram

$$\begin{array}{ccccc} & & z & & \\ & \swarrow & \uparrow & \searrow & \\ x & \xleftarrow{\tau_{\geq 0}z} & & \xrightarrow{0} & y[-1] \\ & \swarrow & \downarrow & \searrow & \\ & & \tau_{\geq 0}z & & \end{array}$$

shows that f is nullhomotopic, which completes the construction of a bounded t -structure on G' , which after the fact is idempotent complete, so $G' \simeq G$.

The inclusion $G \rightarrow \text{Ind}(G)$ is evidently right t -exact with respect to the t -structure defined above on $G' \simeq G$ and the given t -structure on $\text{Ind}(G)$. Let $x \in F_{\leq -1}$. To see left t -exactness, it suffices to check that $\text{Map}_{\text{Ind}(G)}(y, Lx) \simeq 0$ for all $y \in \text{Ind}(G)_{\geq 0}$. But, since $\text{Ind}(G)_{\geq 0}$ is generated under filtered colimits and extensions by images of the objects $z \in F_{\geq 0}$, this result follows from the computation above. Finally, by construction, $F \rightarrow G' \simeq G$ is t -exact. This completes the proof that (i) implies (ii).

Assume (ii). By definition of the t -structure on $\text{Ind}(G)$, the localization functor $L : \text{Ind}(F) \rightarrow \text{Ind}(G)$ is right t -exact. Let $x \in \text{Ind}(F)_{\leq -1}$. We must check that $\text{Map}_{\text{Ind}(G)}(y, Lx) \simeq 0$ for all $y \in \text{Ind}(G)_{\geq 0}$. To do so, it is enough to check this for y of the form Lz for some $z \in F_{\geq 0}$. However, we can write $x \simeq \text{colim}_I x_i$ for a filtered ∞ -category I and some $x_i \in F_{\leq -1}$ since we use the t -structure on $\text{Ind}(F)$ induced by $\text{Ind}(F_{\geq 0})$. Hence,

$$\text{Map}_{\text{Ind}(G)}(Lz, Lx) \simeq \text{colim}_I \text{Map}_{\text{Ind}(G)}(Lz, Lx_i)$$

since L commutes with colimits and Lz is compact in $\text{Ind}(G)$. As $Lz \in G_{\geq 0}$ and $Lx_i \in G_{\leq -1}$, (ii) shows that each mapping space in the colimit on the right is contractible, as desired. Hence, (ii) implies (iii).

To see that (iii) implies (iv), note first that the t -structures on $\text{Ind}(E)$ and $\text{Ind}(F)$ are right complete and hence right separated by Proposition 2.12. It follows from Lemma 2.18 that $\text{Ind}(E)^\heartsuit \subseteq \text{Ind}(F)^\heartsuit$ is

weak Serre. Denote by $i^* : \text{Ind}(E) \rightarrow \text{Ind}(F)$ the induced functor, and let $x \subseteq i^*y$ be a subobject, where $x \in \text{Ind}(F)^\heartsuit$ and $y \in \text{Ind}(E)^\heartsuit$. Then, by t -exactness, $j^*x \subseteq j^*i^*y = 0$, so $j^*x = 0$ in $\text{Ind}(G)^\heartsuit$. It follows that j^*x is in $\text{Ind}(F)^\heartsuit$ and in $\text{Ind}(E)$. Hence, by Lemma 2.15, $x = i^*z$ for some $z \in \text{Ind}(E)^\heartsuit$. Thus, (iv) holds.

Now, suppose that (iv) holds, and let $x \subseteq iy$ for some $y \in E^\heartsuit$ and $x \in F^\heartsuit$. Then, $x \simeq i^*z$ for some $z \in \text{Ind}(E)^\heartsuit$ by hypothesis (iv). However, as an object z of $\text{Ind}(E)$ is compact if and only if i^*z is compact, it follows that in fact $z \in E$. Hence, (iv) implies (i).

The equivalence of (iii) and (v) is [HPV16, Proposition A.5].

Finally, the boundedness of the t -structure on G assuming that (ii) holds follows from the boundedness of the t -structure on F , the essential surjectivity of j up to retracts, and the t -exactness of j . \square

2.3 Excisive squares and adjointability

Consider a commutative square

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array} \quad (1)$$

of small idempotent complete stable ∞ -categories and fully faithful functors. In this section, we establish general conditions (Lemma 2.28, Proposition 2.29, and Theorem 2.30) which guarantee that the induced map

$$\begin{array}{ccc} K(E) & \longrightarrow & K(F) \\ \downarrow & & \downarrow \\ K(G) & \longrightarrow & K(H) \end{array} \quad (2)$$

is a pushout square of spectra and hence gives a long exact sequence

$$\cdots \rightarrow K_n(E) \rightarrow K_n(F) \oplus K_n(G) \rightarrow K_n(H) \rightarrow K_{n-1}(E) \rightarrow \cdots$$

of K -groups. We check these conditions in two situations: for Tate objects (as studied in [Hen16]) later in this section and for t -structures in the proof of Theorem 2.34. We include the former for completeness, while the latter is what we need later in the paper. We begin with a standard lemma about pushouts and cofibers.

Lemma 2.20. *Suppose that*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ P & \xrightarrow{g} & Q \end{array}$$

is a commutative diagram in a stable ∞ -category. Then, the induced map $\text{cofib}(f) \rightarrow \text{cofib}(g)$ is an equivalence if and only if the square is a pushout square.

Proof. If the square is a pushout square, then the horizontal cofibers are equivalent (see [Lur09, Lemma 4.4.2.1]). This is true in any ∞ -category with pushouts and a terminal object. So, assume that $\text{cofib}(f) \rightarrow \text{cofib}(g)$ is an equivalence. Let S be the pushout of P and N over M , and let T be an arbitrary spectrum. Consider

the commutative diagram

$$\begin{array}{ccccc} \mathrm{Map}(\mathrm{cofib}(g), T) & \longrightarrow & \mathrm{Map}(Q, T) & \longrightarrow & \mathrm{Map}(P, T) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}(\mathrm{cofib}(f), T) & \longrightarrow & \mathrm{Map}(S, T) & \longrightarrow & \mathrm{Map}(P, T) \end{array}$$

of fiber sequences of mapping spaces. The outer vertical arrows are equivalences by hypothesis. In general, this does not in general let us conclude that the middle vertical arrow is an equivalence. However, because these are fiber sequences of infinite loop spaces, the long exact sequence in homotopy groups shows that $\mathrm{Map}_{\mathrm{Sp}}(Q, T) \rightarrow \mathrm{Map}_{\mathrm{Sp}}(S, T)$ is an equivalence for all T . Hence, $S \rightarrow Q$ is an equivalence. \square

Let $\widetilde{F/E}$ and $\widetilde{H/G}$ denote the cofibers in $\mathrm{Cat}_{\infty}^{\mathrm{perf}}$ of the horizontal maps in (1). Then, by localization in K -theory, there is a commutative diagram

$$\begin{array}{ccccc} K(E) & \longrightarrow & K(F) & \longrightarrow & K(\widetilde{F/E}) \\ \downarrow & & \downarrow & & \downarrow \\ K(G) & \longrightarrow & K(H) & \longrightarrow & K(\widetilde{H/G}) \end{array} \tag{3}$$

in which the horizontal sequences are cofiber sequences. Hence, using Lemma 2.20, in order to check that (2) is a pushout square it suffices (and is necessary) to see that $K(\widetilde{F/E}) \rightarrow K(\widetilde{H/G})$ is an equivalence. This occurs in particular when $F/E \rightarrow H/G$ is an equivalence after idempotent completion.

Definition 2.21. Say that a square as in (1) is an **excisive square** if $\widetilde{F/E} \rightarrow \widetilde{H/G}$ is an equivalence.

Remark 2.22. It is easy to check using the fully faithfulness of $F/E \rightarrow H/G$ that an excisive square is cartesian, so that $E \rightarrow F \cap G$ is an equivalence.

Example 2.23. (a) If (1) is a pushout square, then it is an excisive square.

(b) Suppose that $E = 0$ and that $H = \langle F, G \rangle$ is a **semiorthogonal decomposition** of H . Recall that this means that F and G are full stable subcategories of H such that

- (i) $F \cap G = 0$,
- (ii) every object $x \in H$ can be written in a cofiber sequence $y \rightarrow x \rightarrow z$ where $y \in G$ and $z \in F$, and
- (iii) the mapping spaces $\mathrm{Map}_H(y, z)$ vanish for all $y \in G$ and all $z \in F$.

Under these conditions, it is easy to check by hand that the induced map $F \rightarrow H/G$ is an equivalence, which induces a (split) localization sequence $K(G) \rightarrow K(H) \rightarrow K(F)$. For more details, see [BGT13].

Remark 2.24. Note that despite conditions (i) and (ii), H is not generally the coproduct in $\mathrm{Cat}_{\infty}^{\mathrm{perf}}$ of F and G . The coproduct is $F \oplus G$, and in that category one has the additional criterion that $\mathrm{Map}_H(z, y) = 0$ for $y \in G$ and $z \in F$. That is, one has an **orthogonal decomposition**. This is a much stronger hypothesis, but it is rarely satisfied in situations of interest. For example, Beilinson's decomposition of $\mathcal{D}^b(\mathbb{P}_k^1) \simeq \langle \mathcal{O}, \mathcal{O}(1) \rangle$ gives a semiorthogonal decomposition of $\mathcal{D}^b(\mathbb{P}_k^1)$ which is not orthogonal (see [Huy06, Corollary 8.29]).

In Proposition 2.29 below, we give a criterion for checking that certain squares (1) are excisive squares. Our arguments are based on those of Benjamin Hennion [Hen16, Proposition 4.2] which in turn are based on those of Sho Saito [Sai15]. We need some preliminaries first.

Definition 2.25 (See [Lur12, Definition 4.7.5.13]). Consider an oriented commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{i^*} & \mathcal{F} \\ p^* \downarrow & & \downarrow q^* \\ \mathcal{G} & \xrightarrow{j^*} & \mathcal{H} \end{array}$$

of ∞ -categories such that i^* and j^* admit right adjoints i_* and j_* , respectively. Fix a natural equivalence $\alpha : j^*p^* \simeq q^*i^*$. The diagram is **right adjointable** if the natural map $p^*i_* \rightarrow j_*j^*p^*i_* \xrightarrow{\alpha} j_*q^*i^*i_* \rightarrow j_*q^*$ is an equivalence.

Remark 2.26. In general, the right adjointability of an oriented diagram as in Definition 2.25 is not equivalent to the adjointability of the transpose diagram.

Proposition 2.27. Consider a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{i} & F \\ p \downarrow & & \downarrow q \\ G & \xrightarrow{j} & H \end{array}$$

of fully faithful functors of stable idempotent complete ∞ -categories. The following conditions are equivalent:

- (1) the induced commutative diagram

$$\begin{array}{ccc} \mathrm{Ind}(F) & \xrightarrow{f^*} & \mathrm{Ind}(F/E) \\ q^* \downarrow & & \downarrow r^* \\ \mathrm{Ind}(H) & \xrightarrow{g^*} & \mathrm{Ind}(H/G) \end{array}$$

of stable presentable ∞ -categories is right adjointable;

- (2) for any $x \in \mathrm{Ind}(F)$, if $i_*x \simeq 0$ in $\mathrm{Ind}(E)$, then $j_*q^*x \simeq 0$ in $\mathrm{Ind}(G)$.

For the proof and the remainder of the section, we will make use of the cofiber sequences $i^*i_*x \rightarrow x \rightarrow f_*f^*x$ in $\mathrm{Ind}(F)$ when $x \in \mathrm{Ind}(F)$ and $\mathrm{Ind}(E) \xrightarrow{i^*} \mathrm{Ind}(F) \xrightarrow{f^*} \mathrm{Ind}(F/G)$ is a localization sequence. In particular, i_* denotes the right adjoint of i^* , notation we will freely throughout the rest of the paper.

Proof. Assume (1). Choose $x \in \mathrm{Ind}(F)$ such that $i_*x \simeq 0$. Then, $x \simeq f_*f^*x$. Now, consider the cofiber sequence

$$j^*j_*q^*f_*f^*x \rightarrow q^*f_*f^*x \rightarrow g_*g^*q^*f_*f^*x \simeq g_*r^*f_*f^*x \simeq g_*r^*f^*x.$$

Adjointability means that the map $q^*f_*f^*x \rightarrow g_*r^*f^*x$ is an equivalence so that $j^*j_*q^*f_*f^*x \simeq 0$. Since j^* is fully faithful, this means that $j_*q^*f_*f^*x \simeq j_*q^*x \simeq 0$, as desired.

We prove (2) implies (1). Let $y \in \mathrm{Ind}(F/E)$. Then, the counit map $f^*f_*y \rightarrow y$ is an equivalence. Set $x = f_*y$. Consider the commutative diagram

$$\begin{array}{ccccc} q^*i^*i_*x & \longrightarrow & q^*x & \longrightarrow & q^*f_*f^*x \\ \downarrow & & \downarrow & & \downarrow \\ j^*j_*q^*x & \longrightarrow & q^*x & \longrightarrow & g_*g^*q^*x \end{array}$$

of cofiber sequences in $\text{Ind}(G)$. Since $i_*x \simeq i_*f_*y \simeq 0$, we have that $j_*q^*x \simeq 0$ by hypothesis (3). Hence, both terms on the left vanish, so the map $q^*f_*f^*x \rightarrow g_*g^*q^*x$ is an equivalence. But, $g_*g^*q^*x \simeq g_*r^*f^*x$. In particular, $g_*r^*y \simeq q^*f_*y$ for all $y \in \text{Ind}(F/E)$. \square

Lemma 2.28. *Suppose that a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{i} & F \\ \downarrow p & & \downarrow q \\ G & \xrightarrow{j} & H \end{array}$$

of fully faithful functors of stable idempotent complete ∞ -categories satisfies the equivalent conditions of Proposition 2.27. Then, $F/E \rightarrow H/G$ is fully faithful.

Proof. We adopt the notation of the proof of the previous proposition. We show that the natural map $\text{Map}_{\text{Ind}(F/E)}(f^*x, f^*y) \rightarrow \text{Map}_{\text{Ind}(H/G)}(r^*f^*x, r^*f^*y)$ is an equivalence for all $x, y \in F/E$. There are natural equivalences,

$$\begin{aligned} \text{Map}_{\text{Ind}(H/G)}(r^*f^*x, r^*f^*y) &\simeq \text{Map}_{\text{Ind}(H/G)}(g^*q^*x, r^*f^*y) \\ &\simeq \text{Map}_{\text{Ind}(H)}(q^*x, g_*r^*f^*y) \\ &\simeq \text{Map}_{\text{Ind}(H)}(q^*x, q^*f_*f^*y) \\ &\simeq \text{Map}_{\text{Ind}(F)}(x, f_*f^*y) \\ &\simeq \text{Map}_{\text{Ind}(F/E)}(f^*x, f^*y), \end{aligned}$$

where the third equivalence is via right adjointability and the fourth follows from the fact that q is fully faithful. \square

Now, we come to an important test for adjointability. We include it for completeness, as it will not be used in the rest of the paper. Rather, when needed, we will check that the equivalent conditions of Proposition 2.27 are satisfied. However, the proof is similar to one step in the proof of Theorem 2.34.

Proposition 2.29. *Let*

$$\begin{array}{ccc} E & \xrightarrow{i} & F \\ \downarrow p & & \downarrow q \\ G & \xrightarrow{j} & H \end{array}$$

be a commutative square of fully faithful functors in $\text{Cat}_\infty^{\text{perf}}$ such that

- (a) *every object y of G is a cofiltered limit $y \simeq \lim_B p(z_\beta)$ such that $jy \simeq \lim_B jpz_\beta$, and*
- (b) *the essential image of q consists of j -cocompact objects of H , meaning that the natural map*

$$\text{colim}_{B^{\text{op}}} \text{Map}_H(jy_\beta, qx) \rightarrow \text{Map}_H(\lim_B jy_\beta, qx)$$

is an equivalence for all $x \in F$ whenever the limit $\lim_B y_\beta$ exists in G and j preserves the limit.

Then, the induced map $F/E \rightarrow H/G$ is fully faithful.

Proof. By Proposition 2.27 and Lemma 2.28, it suffices to prove that $j_*q^*x \simeq 0$ for all $x \in \text{Ind}(F)$ such that $i_*x \simeq 0$.

So, assume that $i_*x \simeq 0$ for some $x \in \text{Ind}(F)$. Note that $j_*q^*x \simeq 0$ if and only if $\text{Map}_{\text{Ind}(G)}(y, j_*q^*x) \simeq 0$ for all $y \in G$. Note also that q^* preserves filtered colimits. Pick one $y \in G$, and use condition (a) to write $y \simeq \lim_B pz_\beta$ where j preserves this limit. If we write $\text{colim}_A x_\alpha \simeq x$ for some filtered ∞ -category A with $x_\alpha \in F$, then, using the compactness of j^*y , there is a chain of equivalences

$$\begin{aligned}
\text{Map}_{\text{Ind}(G)}(y, j_*q^*x) &\simeq \text{Map}_{\text{Ind}(H)}(j^*y, q^*x) \\
&\simeq \text{colim}_A \text{Map}_H(jy, qx_\alpha) \\
&\simeq \text{colim}_A \text{Map}_H(j \lim_B pz_\beta, qx_\alpha) \\
&\simeq \text{colim}_A \text{Map}_H(\lim_B jpz_\beta, qx_\alpha) \\
&\simeq \text{colim}_A \text{colim}_{B^{\text{op}}} \text{Map}_H(jpz_\beta, qx_\alpha) \\
&\simeq \text{colim}_{B^{\text{op}}} \text{colim}_A \text{Map}_F(iz_\beta, x_\alpha) \\
&\simeq \text{colim}_{B^{\text{op}}} \text{Map}_{\text{Ind}(F)}(i^*z_\beta, x) \\
&\simeq \text{colim}_{B^{\text{op}}} \text{Map}_{\text{Ind}(E)}(z_\beta, i_*x) \\
&\simeq 0,
\end{aligned}$$

where we use condition (b) to justify the fifth equivalence. This completes the proof. \square

Theorem 2.30. *Let*

$$\begin{array}{ccc}
E & \xrightarrow{i} & F \\
\downarrow p & & \downarrow q \\
G & \xrightarrow{j} & H
\end{array}$$

be a commutative square of fully faithful functors in $\text{Cat}_\infty^{\text{perf}}$ such that $F/E \rightarrow H/G$ is fully faithful and such that

- (c) every object x of H is a retract of an object x' such that x' fits in to a cofiber sequence $jy \rightarrow x' \rightarrow qz$ for some y in G and some z in F .

Then, the induced square

$$\begin{array}{ccc}
K(E) & \longrightarrow & K(F) \\
\downarrow & & \downarrow \\
K(G) & \longrightarrow & K(H)
\end{array}$$

is a pushout square of spectra.

Proof. By Lemma 2.20, it is enough to show that $K(\widetilde{F/E}) \rightarrow K(\widetilde{H/G})$ is an equivalence, so it is enough to show that $\widetilde{F/E} \rightarrow \widetilde{H/G}$ is an equivalence. By hypothesis this functor is fully faithful, so it is enough to check essential surjectivity. Every object of $\widetilde{H/G}$ is a retract of the image of an object x of H , which is in turn a retract of the image of an object x' of H fitting into a cofiber sequence as in (c). Since $H \rightarrow H/G$ kills jy , it follows that every object of $\widetilde{H/G}$ is a retract of the image of an object of F . Since $\widetilde{F/E}$ is idempotent complete by definition, $\widetilde{F/E} \rightarrow \widetilde{H/G}$ is essentially surjective. \square

Remark 2.31. Note that conditions (a) and (b) in Proposition 2.29 can be used to check fully faithfulness of $F/E \rightarrow H/G$.

Example 2.32. Conditions (a) through (c) of Proposition 2.29 and Theorem 2.30 are meant to abstract the basic property of **Tate objects**. Given a stable idempotent complete ∞ -category E , the ∞ -category $\text{Tate}(E)$ of Tate objects in E fits into a commutative square

$$\begin{array}{ccc} E & \longrightarrow & \text{Ind}(E) \\ \downarrow & & \downarrow \\ \text{Pro}(E) & \longrightarrow & \text{Tate}(E), \end{array}$$

and this square satisfies the properties of the theorem (for ease of exposition, we suppress set-theoretic issues and refer the reader to [Hen16] for a careful treatment). To check condition (a), note that *every* object of $\text{Pro}(E)$ can be written as a cofiltered limit, and $\text{Pro}(E) \rightarrow \text{Tate}(E)$ preserves cofiltered limits by the universal property of Tate objects (see [Hen16, Theorem 2.6]). Condition (b) follows from the fact that there is a natural embedding $\text{Tate}(E) \rightarrow \text{Pro Ind}(E)$ which preserves cofiltered limits by definition of the mapping spaces in a pro-category. Condition (c) follows for example from [Hen16, Corollary 3.4].

The key point about the ∞ -category of Tate objects is that, by the theorem, $K(\text{Tate}(E)) \simeq \Sigma K(E)$. Indeed, $K(\text{Ind}(E)) \simeq 0 \simeq K(\text{Pro}(E))$ because of the existence of countable (co)products, which means that

$$\begin{array}{ccc} K(E) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K(\text{Tate}(E)) \end{array}$$

is a pushout square.

Remark 2.33. It is possible to build an ∞ -category $\text{Tate}^\kappa(E)$ of κ -Tate objects out of $\text{Ind}(E)^\kappa$ and ${}^\kappa\text{Pro}(E)$, the full subcategory of κ -cocompact objects in $\text{Pro}(E)$. This construction has the same properties as $\text{Tate}(E)$ but has the advantage that it is small and hence does not require working in a larger universe. Such an approach is closer to the spirit of this paper and is done for exact categories in [BGW14].

2.4 Vanishing of K_{-1}

We prove our analogue of Schlichting's theorem [Sch06, Theorem 6] in the case of a stable ∞ -category admitting a bounded t -structure. The proof differs substantially from that of Schlichting.

Theorem 2.34. *If E is a small stable ∞ -category with a bounded t -structure, then $K_{-1}(E) = 0$.*

Proof. The t -structure on E extends to a t -structure on $\text{Ind}(E)$ with nonnegative objects $\text{Ind}(E_{\geq 0}) \simeq \text{Ind}(E)_{\geq 0}$ by Proposition 2.12. Let $A = E^\heartsuit$ denote the heart of E , and fix κ an uncountable regular cardinal such that E is essentially κ -small. Consider the commutative diagram of fully faithful functors

$$\begin{array}{ccc} E & \xrightarrow{i} & \text{Ind}_A^+(E) \\ p \downarrow & & \downarrow q \\ \text{Ind}_A^-(E)^\kappa & \xrightarrow{j} & \text{Ind}_A(E)^\kappa, \end{array} \tag{4}$$

where

- $\text{Ind}_A^+(E) \subseteq \bigcup_n \text{Ind}(E)_{\leq n} \simeq \bigcup_n \text{Ind}(E)_{\leq n}$ is the full subcategory of bounded above objects x with $\pi_n x \in A$ for all n ,
- $\text{Ind}_A^-(E)^\kappa \subseteq \bigcup_n \text{Ind}(E_{\geq n})^\kappa$ is the full subcategory of the κ -compact bounded below objects with $\pi_n(x) \in A$ for all n , and
- $\text{Ind}_A(E)^\kappa$ is the full subcategory of $\text{Ind}(E)$ of objects $x \in \text{Ind}(E)$ such that $\tau_{\leq n} x \in \text{Ind}_A^+(E)$ and $\tau_{\geq n} x \in \text{Ind}_A^-(E)^\kappa$ for all n .

Note that the inclusion $\text{Ind}_A^+(E) \rightarrow \text{Ind}(E)$ factors through $\text{Ind}_A(E)^\kappa$. Indeed, for $x \in \text{Ind}_A^+(E)$, the truncations $\tau_{\leq n} x$ are in $\text{Ind}_A^+(E)$ for all n . Moreover, since $\tau_{\geq n} x$ is in fact in E , it follows that $\tau_{\geq n} x \in \text{Ind}_A^-(E)^\kappa$.

The objects of $\text{Ind}_A^-(E)^\kappa$ are in fact κ -compact in $\text{Ind}(E)$ because $\text{Ind}(E_{\geq 0}) \rightarrow \text{Ind}(E)$ preserves κ -compact objects as the right adjoint preserves (ω) -filtered colimits, and hence all κ -filtered colimits. For the same reason, $\text{Ind}(E_{\geq n}) \rightarrow \text{Ind}(E_{\geq n-1})$ preserves κ -compact objects. Clearly, if $x \in \text{Ind}_A(E)^\kappa$, then every truncation $\tau_{\geq n} x$ is κ -compact, however we do not claim that every κ -compact object x in $\text{Ind}(E)$ with $\pi_n(x) \in A$ is contained in $\text{Ind}_A(E)^\kappa$.

We do claim that

- (1) $\text{Ind}_A(E)^\kappa$, $\text{Ind}_A^+(E)$, and $\text{Ind}_A^-(E)^\kappa$ are essentially small idempotent complete stable subcategories of $\text{Ind}(E)$ and
- (2) that the t -structure on $\text{Ind}(E)$ restricts to a t -structure on $\text{Ind}_A(E)^\kappa$.

After establishing these facts, we prove that we can apply Theorem 2.30 to the square (4). This gives a pushout square of K -theory spectra which lets us prove in the end that $K_{-1}(E) = 0$.

In fact $\text{Ind}_A(E)^\kappa$ is contained in $\text{Ind}(E)^\kappa$. Since the bounded below objects of $\text{Ind}_A(E)^\kappa$ are κ -compact, it is enough to check that $\text{Ind}_A^+(E) \subseteq \text{Ind}(E)^\kappa$. Let $x \in \text{Ind}_A^+(E)$, so that in particular $x \in \text{Ind}(E)_{\leq n}$ for some n . There are maps

$$\pi_n x[n] \simeq \tau_{\geq n} x \rightarrow \tau_{\geq n-1} x \rightarrow \tau_{\geq n-2} x \rightarrow \cdots \rightarrow x. \quad (5)$$

Since the induced t -structure on $\text{Ind}(E)$ is right complete by Proposition 2.12, the colimit of the sequence is equivalent to x . To see this, note that it is enough to prove that $\text{colim}_i \text{Map}(y, \tau_{\geq n-i} x) \simeq \text{Map}(y, x)$ for all $y \in E$. However, since the t -structure on E is bounded, any such y is contained in $E_{\geq n-i}$ for some i . Using the two cofiber sequences

$$\tau_{\geq n-i} x \rightarrow \tau_{\geq n-j} x \rightarrow \tau_{\leq n-i-1} \tau_{\geq n-j} x$$

and

$$\tau_{\geq n-i} x \rightarrow x \rightarrow \tau_{\leq n-i-1} x,$$

for $j \geq i$, we see that

$$\text{Map}(y, \tau_{\geq n-i} x) \simeq \text{Map}(y, \tau_{\geq n-j} x)$$

and

$$\text{Map}(y, \tau_{\geq n-i} x) \simeq \text{Map}(y, x)$$

for $j \geq i$. This proves that the colimit of (5) is indeed x . However, each object $\tau_{\geq m} x$ is actually in E , so this is a κ -small colimit of compact objects and hence of κ -compact objects in $\text{Ind}(E)$. Thus, x is κ -compact by [Lur09, Corollary 5.3.4.15].

That these three ∞ -categories are essentially small follows from the fact that $\text{Ind}_A(E)^\kappa \subseteq \text{Ind}(E)^\kappa$ and the fact that $\text{Ind}(E)^\kappa$ is essentially small because every object is the colimit in presheaves on E of a κ -small diagram [Lur09, Proposition 5.3.4.17]. Moreover, $\text{Ind}_A^+(E)$ is idempotent complete because $\text{Ind}(E)_{\leq 0}$ and A are idempotent complete, while each $\text{Ind}(E_{\geq n})^\kappa$ is idempotent complete since it is closed under κ -small

colimits, and in particular it is closed under idempotent completion because κ is uncountable. It follows that $\text{Ind}_A(E)^\kappa$ is idempotent complete as well since inclusion in $\text{Ind}_A(E)^\kappa$ is given by a condition on the truncations.

These three ∞ -categories are closed under suspension and desuspension in $\text{Ind}(E)$, so to see that they are stable, it is enough to show that they are closed under either taking fibers or cofibers in $\text{Ind}(E)$ by [Lur12, Lemma 1.1.3.3] and its opposite version. We first note that if z is the cofiber of a map $f : x \rightarrow y$ in $\text{Ind}(E)$ between two objects such that $\pi_n x$ and $\pi_n y$ are in A for all n , then $\pi_n z$ is an extension of objects of A , namely of $\ker(\pi_{n+1}x \rightarrow \pi_{n+1}y)$ by $\text{coker}(\pi_n x \rightarrow \pi_n y)$. Since E^\heartsuit is closed under extension in $\text{Ind}(E)^\heartsuit$, by Lemma 2.18, we see that $\pi_n z \in A$.

Stability of $\text{Ind}_A^+(E)$ follows from the fact the cofiber of a map of bounded above objects is bounded above; stability of $\text{Ind}_A^-(E)^\kappa$ follows from the fact that $\text{Ind}(E_{\geq n})^\kappa$ is closed under cofibers in $\text{Ind}(E)$.

We show that $\text{Ind}_A(E)^\kappa$ is stable, by showing that it is closed under taking fibers in $\text{Ind}(E)$. If $z \rightarrow x \rightarrow y$ is a fiber sequence in $\text{Ind}(E)$ where $x \rightarrow y$ is in $\text{Ind}_A(E)^\kappa$, then $\pi_n z \in A$ for all n , so that $\tau_{\leq n} z \in \text{Ind}_A^+(E)$ for all n . Hence, it is enough to show that $\tau_{\geq n} z \in \text{Ind}(E_{\geq n})^\kappa$. However, $\tau_{\geq n} : \text{Ind}(E) \rightarrow \text{Ind}(E_{\geq n})$ preserves limits as it is a right adjoint. Hence, $\tau_{\geq n} z$ is the fiber of $\tau_{\geq n} x \rightarrow \tau_{\geq n} y$ in $\text{Ind}(E_{\geq 0})$. Now, as we have chosen κ to be uncountable and such that E is essentially κ -small, we see by [Lur09, Proposition 5.4.7.4] that the inclusion $\text{Ind}(E_{\geq n})^\kappa \rightarrow \text{Ind}(E_{\geq n})$ is closed under all finite limits in $\text{Ind}(E_{\geq n})$, and in particular under fibers. Hence, $z \in \text{Ind}_A(E)^\kappa$, which completes the proof of claim (1).

Suppose that $x \in \text{Ind}_A(E)^\kappa$. To show that the t -structure on $\text{Ind}(E)$ restricts to $\text{Ind}_A(E)^\kappa$, we show that $\tau_{\geq n} x$ and $\tau_{\leq n} x$ are in $\text{Ind}_A(E)^\kappa$. In fact, by stability, it is sufficient to check only one of these. Moreover, $\tau_{\geq m} \tau_{\geq n} x \simeq \tau_{\geq m} x$ for $m \geq n$, so that if $x \in \text{Ind}_A^-(E)^\kappa$, then so is $\tau_{\geq m} x$ for all m . Similarly, $\tau_{\leq m} \tau_{\geq n} x \in E \subseteq \text{Ind}_A^+(E)$. So, $\tau_{\geq m} x \in \text{Ind}_A^-(E)$, which proves that $\text{Ind}_A(E)^\kappa$ inherits the induced t -structure from $\text{Ind}(E)$. This proves (2). Note that by construction the truncation functors on $\text{Ind}_A(E)^\kappa$ preserve $\text{Ind}_A^+(E)$ and $\text{Ind}_A^-(E)^\kappa$, which therefore inherit compatible t -structures.

To complete the proof, we will show that the square (4) satisfies the hypotheses of Theorem 2.30. The validity of condition (c) in the theorem is due to the t -structure, which gives cofiber sequence $\tau_{\geq 0} x \rightarrow x \rightarrow \tau_{\leq -1} x$ for every $x \in \text{Ind}_A(E)^\kappa$, where $\tau_{\geq 0} x \in \text{Ind}_A^-(E)^\kappa$ and $\tau_{\leq -1} x \in \text{Ind}_A^+(E)$. To prove that the induced functor

$$\text{Ind}_A^+(E)/E \rightarrow \text{Ind}_A(E)^\kappa/\text{Ind}_A^-(E)^\kappa$$

is fully faithful, we will prove directly that the square satisfies Proposition 2.27(2) and invoke Lemma 2.28.

Let $x \in \text{Ind}(\text{Ind}_A^+(E))$. We need to show that if $i_* x \simeq 0$ then $j_* q^* x \simeq 0$. It is enough to prove that $\text{Map}_{\text{Ind}(\text{Ind}_A^-(E)^\kappa)}(y, j_* q^* x) \simeq 0$ for $y \in \text{Ind}_A^-(E)^\kappa$. Choose a filtered ∞ -category B and an equivalence $\text{colim}_B x_\beta \simeq x$ where $x_\beta \in \text{Ind}_A^+(E)$ for all β in B . Using adjunctions and compactness, we get a chain of

equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Ind}(\mathrm{Ind}_A^-(E)^\kappa)}(y, j_* q^* x) &\simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Ind}_A(E)^\kappa)}(j^* y, q^* x) \\
&\simeq \mathrm{colim}_B \mathrm{Map}_{\mathrm{Ind}(E)^\kappa}(j y, q x_\beta) \\
&\simeq \mathrm{colim}_B \mathrm{Map}_{\mathrm{Ind}(E)}(y, x_\beta) \\
&\simeq \mathrm{colim}_B \mathrm{colim}_n \mathrm{Map}_{\mathrm{Ind}(E)}(\tau_{\leq n} y, x_\beta) \\
&\simeq \mathrm{colim}_B \mathrm{colim}_n \mathrm{Map}_{\mathrm{Ind}_A^+(E)}(\tau_{\leq n} y, x_\beta) \\
&\simeq \mathrm{colim}_n \mathrm{colim}_B \mathrm{Map}_{\mathrm{Ind}_A^+(E)}(\tau_{\leq n} y, x_\beta) \\
&\simeq \mathrm{colim}_n \mathrm{Map}_{\mathrm{Ind}(\mathrm{Ind}_A^+(E))}(i^* \tau_{\leq n} y, x) \\
&\simeq \mathrm{colim}_n \mathrm{Map}_{\mathrm{Ind}(E)}(\tau_{\leq n} y, i_* x) \\
&\simeq 0,
\end{aligned}$$

where we use (i) the crucial fact that x_β is bounded above as well as the t -structure to observe that the colimit $\mathrm{colim}_{n \rightarrow \infty} \mathrm{Map}_{\mathrm{Ind}(E)}(\tau_{\leq n} y, x_\beta)$ stabilizes at $\mathrm{Map}_{\mathrm{Ind}(E)}(y, x_\beta)$ and (ii) that $\tau_{\leq n} j y$ is in $E \subseteq \mathrm{Ind}_A^+(E)$ and hence $i^* \tau_{\leq n} j y$ is compact in $\mathrm{Ind}(\mathrm{Ind}_A^+(E))$ to prove the eighth equivalence.

It follows that there is a cofiber sequence

$$K(E) \rightarrow K(\mathrm{Ind}_A^-(E)^\kappa) \oplus K(\mathrm{Ind}_A^+(E)) \rightarrow K(\mathrm{Ind}_A(E)^\kappa).$$

of K -theory spectra. It is easy to see that the K -theory spectra of the idempotent complete stable ∞ -categories $\mathrm{Ind}_A^+(E)$ and $\mathrm{Ind}_A^-(E)^\kappa$ are zero. Indeed, there is an endofunctor $T : \mathrm{Ind}_A^-(E)^\kappa \rightarrow \mathrm{Ind}_A^-(E)^\kappa$ given by $T = \bigoplus_{n \geq 0} \mathrm{id}[2n]$ since $\mathrm{Ind}(E)^\kappa$ is closed under countable coproducts. We have an equivalence of endofunctors $T \simeq \mathrm{id} \oplus T[2]$. Hence, the identity map on $K(\mathrm{Ind}_A^-(E)^\kappa)$ is nullhomotopic. The same argument but with desuspensions shows that $K(\mathrm{Ind}_A^+(E)) \simeq 0$. Hence,

$$K_0(\mathrm{Ind}_A(E)^\kappa) \cong K_{-1}(E).$$

Given an object x of $\mathrm{Ind}_A(E)^\kappa$, we have a canonical triangle

$$\tau_{\geq 0} x \rightarrow x \rightarrow \tau_{\leq -1} x$$

coming from the t -structure, where $\tau_{\geq 0} x$ is in $\mathrm{Ind}_A^-(E)^\kappa$ and $\tau_{\leq -1} x$ is in $\mathrm{Ind}_A^+(E)$. But, since K_0 of each of the half-bounded categories is zero, it follows that the class of x in $K_{-1}(E)$ is also zero. \square

3 Induction

This section contains the proofs of the inductive step of our main theorem and the nonconnective theorem of the heart in the noetherian case, their relation to the Farrell-Jones conjecture in negative K -theory for group rings, and a discussion of the major impediments to proving the conjecture in general.

3.1 Dualizability of compactly generated stable ∞ -categories

We discuss in this section some technical preliminaries about dualizability we will need later. The material here is basically well-known, but we include it for the sake of completeness.

Recall that an object x in a symmetric monoidal ∞ -category \mathcal{P} is **dualizable** if there is another object, Dx together with an evaluation map $\text{ev} : x \otimes Dx \rightarrow \mathbb{1}$ and a coevaluation map $\text{coev} : \mathbb{1} \rightarrow Dx \otimes x$ such that the composites

$$x \xrightarrow{\text{id}_x \otimes \text{coev}} x \otimes Dx \otimes x \xrightarrow{\text{ev} \otimes \text{id}_x} x$$

and

$$Dx \xrightarrow{\text{coev} \otimes \text{id}_{Dx}} Dx \otimes x \otimes Dx \xrightarrow{\text{id}_{Dx} \otimes \text{ev}} Dx$$

are equivalent to the identities on x and Dx , respectively.

In a closed symmetric monoidal ∞ -category \mathcal{P} , the endofunctor induced by tensoring with a fixed object x has a right adjoint taking y to y^x by definition. Tensoring with x has a left adjoint if and only if x is dualizable, in which case the unit and counit maps of the adjunction are given by tensoring with coev and ev , respectively. Moreover, when x is a dualizable object in the closed symmetric monoidal ∞ -category \mathcal{P} , there is a natural equivalence $y \otimes x \simeq y^{Dx}$ for $y \in \mathcal{P}$.

Proposition 3.1. *If \mathcal{C} be a compactly generated stable ∞ -category, then \mathcal{C} is dualizable in $\text{Pr}_{\text{st}}^{\text{L}}$ with dual $\text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp})$.*

Proof. We refer to [Lur12, Section 4.8] for information about the tensor product of stable presentable ∞ -categories. Because colimits in $\text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp})$ are computed pointwise, the evaluation bifunctor $\mathcal{C} \times \text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp}) \rightarrow \text{Sp}$ preserves colimits separately in each variable, so we obtain an evaluation map $\mathcal{C} \otimes \text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp}) \rightarrow \text{Sp}$. We must define a coevaluation map $\text{Sp} \rightarrow \text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp}) \otimes \mathcal{C}$, which is to say an object of

$$\text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp}) \otimes \mathcal{C} \simeq \text{Fun}^{\text{lim}}(\mathcal{C}^{\text{op}}, \text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp})),$$

where Fun^{lim} denotes the ∞ -category of limit-preserving functors. Using the fact that \mathcal{C} is stable and compactly generated (i.e. $\mathcal{C} \simeq \text{Ind}(\mathcal{C}^{\omega})$) we have an equivalence $\text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp}) \simeq \text{Fun}^{\text{ex}}(\mathcal{C}^{\omega}, \text{Sp})$. Moreover, the (restricted) spectral co-Yoneda embedding $h : \mathcal{C}^{\text{op}} \rightarrow \text{Fun}(\mathcal{C}^{\omega}, \text{Sp})$ preserves limits and factors through the full subcategory $\text{Fun}^{\text{ex}}(\mathcal{C}^{\omega}, \text{Sp}) \subseteq \text{Fun}(\mathcal{C}^{\omega}, \text{Sp})$. This gives the desired limit-preserving functor $\mathcal{C}^{\text{op}} \rightarrow \text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp})$. It is then routine to verify the triangle identities, so that \mathcal{C} is dualizable with dual $\text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp})$. For example, consider the composition

$$\mathcal{C} \xrightarrow{\text{id}_{\mathcal{C}} \otimes \text{coev}} \mathcal{C} \otimes \text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp}) \otimes \mathcal{C} \xrightarrow{\text{ev} \otimes \text{id}_{\mathcal{C}}} \mathcal{C},$$

which we can write as the composition

$$\mathcal{C} \rightarrow \text{Fun}^{\text{lim}}(\mathcal{C}^{\text{op}}, \mathcal{C} \otimes \text{Fun}^{\text{L}}(\mathcal{C}, \text{Sp})) \xrightarrow{\text{ev}} \text{Fun}^{\text{lim}}(\mathcal{C}^{\text{op}}, \text{Sp}).$$

By definition of the coevaluation map, the composition is the Yoneda embedding $\mathcal{C} \rightarrow \text{Fun}^{\text{lim}}(\mathcal{C}^{\text{op}}, \text{Sp})$. Since the natural equivalence $\text{Fun}^{\text{lim}}(\mathcal{C}^{\text{op}}, \text{Sp}) \simeq \mathcal{C}$ takes the representable functor $h(x)$ to x , we see that the composition is equivalent to the identity. The argument for the dual is similar. \square

Lemma 3.2. *If \mathcal{C} and \mathcal{D} are compactly generated stable ∞ -categories, then $\mathcal{C} \otimes \mathcal{D}$ is compactly generated by objects the $x \otimes y$, where x and y range over compact generators of \mathcal{C} and \mathcal{D} , respectively.*

Proof. We show first that these objects are compact. The mapping spectrum functor

$$(\mathcal{C} \otimes \mathcal{D})(x \otimes y, -) : \mathcal{C} \otimes \mathcal{D} \rightarrow \text{Mod}_{\mathbb{S}}$$

preserves filtered colimits if and only if it admits a right adjoint by the adjoint functor theorem, and when this is the case it represents an object in $\text{Fun}^{\text{L}}(\mathcal{C} \otimes \mathcal{D}, \text{Mod}_{\mathbb{S}})$. By the universal property of $\mathcal{C} \otimes \mathcal{D}$ this occurs if and only if $\mathcal{C}(x, -) \otimes_{\mathbb{S}} \mathcal{D}(y, -) : \mathcal{C} \times \mathcal{D} \rightarrow \text{Mod}_{\mathbb{S}}$ preserves filtered colimits in each variable, using the fact that if $x, x' \in \mathcal{C}^{\omega}$ and $y, y' \in \mathcal{D}^{\omega}$, then

$$(\mathcal{C} \otimes \mathcal{D})(x \otimes y, x' \otimes y') \simeq \mathcal{C}(x, x') \otimes_{\mathbb{S}} \mathcal{D}(y, y').$$

This happens if and only if x and y are both compact.

By definition, $\mathcal{C} \otimes \mathcal{D}$ is the universal stable presentable ∞ -category equipped with a functor $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \otimes \mathcal{D}$ preserving small colimits in each variable (see [Lur09, Remark 5.5.3.9]). Suppose that $F : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ is a functor preserving colimits in each variable such that $F(x, y) = 0$ for all $x \in \mathcal{C}^\omega$ and $y \in \mathcal{D}^\omega$. By definition of F and the fact that every object of \mathcal{C} (resp. \mathcal{D}) can be written as a small colimit of objects of \mathcal{C}^ω (resp. \mathcal{D}^ω), it follows that F vanishes. It follows that the objects

$$\{x \otimes y : x \in \mathcal{C}^\omega, y \in \mathcal{D}^\omega\}$$

generate $\mathcal{C} \otimes \mathcal{D}$. □

Proposition 3.3. *Let \mathcal{C} be dualizable object of $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$. Then for any fully faithful functor $\mathcal{A} \rightarrow \mathcal{B}$ in Pr^{L} , the induced functor $\mathcal{A} \otimes \mathcal{C} \rightarrow \mathcal{B} \otimes \mathcal{C}$ is fully faithful.*

Proof. Let \mathcal{D} be a dual of \mathcal{C} . We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{A} \otimes \mathcal{C} & \longrightarrow & \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{A}) & \longrightarrow & \mathrm{Fun}(\mathcal{D}, \mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} \otimes \mathcal{C} & \longrightarrow & \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{B}) & \longrightarrow & \mathrm{Fun}(\mathcal{D}, \mathcal{B}), \end{array}$$

in which the left hand horizontal maps are equivalences and the right hand horizontal maps are fully faithful. (Technically, $\mathrm{Fun}(\mathcal{D}, -)$ lands in a higher universe, but we can restrict to the κ -continuous functors for any κ such that \mathcal{D} is κ -compactly generated.) Moreover, the right hand vertical map is fully faithful since $\mathcal{A} \rightarrow \mathcal{B}$ is, by hypothesis, so it follows that each of the other vertical maps is fully faithful. □

Note that we did not actually use the fact that \mathcal{C} was stable in the proof of the above proposition; the same argument works for \mathcal{C} dualizable in Pr^{L} . Unfortunately, there are not so many dualizable objects of Pr^{L} , but as soon as we pass to $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$, we obtain a vast supply by Proposition 3.1.

A **localization sequence** in $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ is a cofiber sequence $\mathcal{B} \rightarrow \mathcal{C} \rightarrow \mathcal{D}$ such that $\mathcal{B} \rightarrow \mathcal{C}$ is fully faithful. The stable presentable ∞ -category \mathcal{D} in this case is equivalent to the usual Bousfield localization of \mathcal{C} at the arrows with cofiber in \mathcal{B} by [BGT13, Proposition 5.6].

Example 3.4. The prototypical localization sequence arises from a quasi-compact and quasi-separated scheme X together with a quasi-compact open subscheme $U \subseteq X$ with complement Z . In this case, the functor $\mathcal{D}(X) \rightarrow \mathcal{D}(U)$ is a localization with kernel $\mathcal{D}_Z(X)$, the ∞ -category of complexes of \mathcal{O}_X -modules with quasi-coherent cohomology sheaves supported set-theoretically on Z . Hence, $\mathcal{D}_Z(X) \rightarrow \mathcal{D}(X) \rightarrow \mathcal{D}(U)$ is a localization sequence.

Since localization sequences in $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ are cofiber sequences and as the tensor product on $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ preserves small colimits in each variable by [Lur12, Remark 4.8.1.23], the previous lemma shows that localization sequences of stable presentable ∞ -categories are preserved by tensoring with a given compactly generated stable ∞ -category \mathcal{E} . Thus, we have proved the following.

Corollary 3.5. *Let $\mathcal{B} \rightarrow \mathcal{C} \rightarrow \mathcal{D}$ be a localization sequence of stable presentable ∞ -categories. Then,*

$$\mathcal{B} \otimes \mathcal{E} \rightarrow \mathcal{C} \otimes \mathcal{E} \rightarrow \mathcal{D} \otimes \mathcal{E}$$

is a localization sequence for any compactly generated stable ∞ -category \mathcal{E} .

3.2 Negative K -theory via ∞ -categories of automorphisms

In this section, we prove the following theorem, which verifies Conjecture B in many cases.

Theorem 3.6. *If E is a small stable ∞ -category equipped with a bounded t -structure such that E^\heartsuit is noetherian, then $K_{-n}(E) = 0$ for $n \geq 1$.*

Many of our arguments in the proof work in greater generality, and we take care to isolate those parts that are truly special to the situation of a noetherian heart.

Definition 3.7. Throughout this section, $\mathbb{S}[s] = \Sigma_+^\infty \mathbb{N}$ denotes the free commutative \mathbb{S} -algebra on the commutative monoid \mathbb{N} . Note that $\mathbb{S}[s]$ is equivalent to the free \mathbb{E}_1 -ring spectrum on the sphere spectrum \mathbb{S} . Similarly, $\mathbb{S}[s^{\pm 1}] = \Sigma_+^\infty \mathbb{Z}$ is the free commutative \mathbb{S} -algebra on the commutative monoid \mathbb{Z} , or, equivalently, the localization of $\mathbb{S}[s]$ obtained by inverting $s \in \pi_0 \mathbb{S}[s]$. These are each flat over \mathbb{S} and have the expected ring of components; that is, $\pi_0(\mathbb{S}[s]) \cong \mathbb{Z}[s]$ and $\pi_0(\mathbb{S}[s^{\pm 1}]) \cong \mathbb{Z}[s^{\pm 1}]$, while $\pi_*(\mathbb{S}[s]) \cong (\pi_* \mathbb{S})[s]$ and $\pi_*(\mathbb{S}[s^{\pm 1}]) \cong (\pi_* \mathbb{S})[s^{\pm 1}]$.

Notation 3.8. In general, we will use either the notation Mod_R or $\mathcal{D}(R)$ for the stable presentable ∞ -category of right R -modules for an \mathbb{E}_∞ -ring spectrum R . Moreover, in the special cases of $\mathbb{S}[s]$ and $\mathbb{S}[s^{\pm 1}]$, we will use the suggestive notation $\mathcal{D}(\mathbb{A}^1) = \text{Mod}_{\mathbb{S}[s]}$ and $\mathcal{D}(\mathbb{G}_m) = \text{Mod}_{\mathbb{S}[s^{\pm 1}]}$. Given a stable presentable ∞ -category \mathcal{C} , we write

$$\mathcal{D}(\mathbb{A}^1, \mathcal{C}) = \mathcal{D}(\mathbb{A}^1) \otimes \mathcal{C} = \text{Mod}_{\mathbb{S}[s]} \otimes \mathcal{C},$$

and similarly for $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$.

To begin, we show that $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ can be identified with the ∞ -category of endomorphisms in \mathcal{C} , and that $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$ is equivalent to the ∞ -category of automorphisms in \mathcal{C} .

Definition 3.9. Given an ∞ -category \mathcal{C} , the functor category

$$\text{Fun}(\Delta^1 / \partial \Delta^1, \mathcal{C})$$

is the ∞ -category of endomorphisms in \mathcal{C} . An object of the ∞ -category of endomorphisms consists of a pair (x, e) where x is an object of \mathcal{C} and $e : x \rightarrow x$ is an endomorphism. For example, if \mathcal{C} is additive, then $(x, 0)$ (the object x equipped with the zero endomorphism) and (x, id_x) are functorial sections of the forgetful functor $\text{Fun}(\Delta^1 / \partial \Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$. The ∞ -category of automorphisms in \mathcal{C} is

$$\text{Fun}(S^1, \mathcal{C}),$$

where $S^1 \simeq B\mathbb{Z}$ is a Kan complex weakly equivalent to $\Delta^1 / \partial \Delta^1$. The map of ∞ -categories $\Delta^1 / \partial \Delta^1 \rightarrow S^1$ induces a fully faithful embedding

$$\text{Fun}(S^1, \mathcal{C}) \rightarrow \text{Fun}(\Delta^1 / \partial \Delta^1, \mathcal{C})$$

with essential image those endomorphisms (x, e) such that $e : x \rightarrow x$ is an equivalence.

Proposition 3.10. *If \mathcal{C} is a stable presentable ∞ -category, then*

- (i) $\mathcal{D}(\mathbb{A}^1, \mathcal{C}) \simeq \text{Fun}(\Delta^1 / \partial \Delta^1, \mathcal{C})$, and
- (ii) $\mathcal{D}(\mathbb{G}_m, \mathcal{C}) \simeq \text{Fun}(S^1, \mathcal{C})$.

Proof. We prove (i), the proof of (ii) being similar. We claim that there is a natural equivalence

$$\mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathcal{C}) \simeq \mathrm{Fun}^L(\mathrm{Mod}_{\mathbb{S}[s]}, \mathcal{C}).$$

It suffices to show that $\mathrm{Mod}_{\mathbb{S}[s]}$ is the free stable presentable ∞ -category generated by $\Delta^1/\partial\Delta^1$. This follows from the $(\mathrm{Mod}_*, \mathrm{End})$ adjunction [AG14, Section 3.1] together with the fact that $\mathbb{S}[s] \simeq \mathbb{S}[\mathbb{N}]$ is the free \mathbb{S} -algebra on the monoid \mathbb{N} , and that the nerve of \mathbb{N} (viewed as a category with one object) is a fibrant replacement for $\Delta^1/\partial\Delta^1$ in the Joyal model structure. For any \mathbb{S} -algebra R and any stable presentable ∞ -category \mathcal{C} , there is a natural equivalence $\mathrm{Mod}_{R^{\mathrm{op}}} \otimes \mathcal{C} \simeq \mathrm{Fun}^L(\mathrm{Mod}_R, \mathcal{C})$. Indeed, Mod_R is compactly generated, and hence dualizable by Proposition 3.1 with dual $\mathrm{Mod}_{R^{\mathrm{op}}}$. In particular, since $\mathbb{S}[s]$ is an \mathbb{E}_∞ -ring spectrum, the ∞ -category of endomorphisms in a stable presentable ∞ -category \mathcal{C} is equivalent to $\mathrm{Mod}_{\mathbb{S}[s]} \otimes \mathcal{C}$. \square

We focus now on the case where $\mathcal{C} \simeq \mathrm{Ind}(E)$ is compactly generated by a small stable ∞ -category E .

Lemma 3.11. *If $\mathcal{C} \simeq \mathrm{Ind}(E)$ is compactly generated, then $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ is compactly generated by the objects $\mathbb{S}[s] \otimes x =: x[s]$ and $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$ is compactly generated by the objects $\mathbb{S}[s^{\pm 1}] \otimes x =: x[s^{\pm 1}]$ as x ranges over the objects of E .*

Proof. This is a special case of Lemma 3.2. \square

Lemma 3.12. *If $\mathcal{C} \simeq \mathrm{Ind}(E)$ is compactly generated, then the natural functor $\mathcal{D}(\mathbb{A}^1, \mathcal{C}) \rightarrow \mathcal{D}(\mathbb{G}_m, \mathcal{C})$ is a localization with kernel a compactly generated stable presentable ∞ -category which we will denote $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$. Moreover, $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$ is compactly generated by the compact objects $(x, 0)$ in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ as x ranges over the objects of E .*

The fact that $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$ is generated by compact objects that are compact in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ implies that the right adjoint $\mathcal{D}(\mathbb{G}_m, \mathcal{C}) \rightarrow \mathcal{D}(\mathbb{A}^1, \mathcal{C})$ to the localization preserves filtered colimits.

Proof. By [Lur12, Proposition 7.2.4.17] or [AG14, Proposition 6.9], we have a localization sequence

$$\mathrm{Mod}_{\mathbb{A}^1, \{0\}} \rightarrow \mathrm{Mod}_{\mathbb{A}^1} \rightarrow \mathrm{Mod}_{\mathbb{G}_m}$$

of stable presentable ∞ -categories, where $\mathrm{Mod}_{\mathbb{A}^1, \{0\}}$ is the full subcategory of $\mathrm{Mod}_{\mathbb{A}^1}$ consisting of $\mathbb{S}[s]$ -modules M such that for every $x \in \pi_m(M)$ there exists a positive integer N such that $s^N \cdot x = 0$. Moreover, by [AG14, Proposition 6.9], $\mathrm{Mod}_{\mathbb{A}^1, \{0\}}$ is compactly generated by the object \mathbb{S} when viewed as an $\mathbb{S}[s]$ -module; in particular, since $\mathbb{S} \simeq \mathrm{cofib}(\mathbb{S}[s] \xrightarrow{t} \mathbb{S}[s])$ is compact as an $\mathbb{S}[s]$ -module, $\mathrm{Mod}_{\mathbb{A}^1, \{0\}}$ is compactly generated by compact objects of $\mathrm{Mod}_{\mathbb{A}^1}$. By tensoring with $\mathrm{Ind}(E)$, we obtain the localization sequence we want by Corollary 3.5. The object $\mathbb{S} \otimes x$ is by definition x with the zero endomorphism. \square

We turn to the problem of constructing t -structures on ∞ -categories of endomorphisms and automorphisms.

Lemma 3.13. *Let $\mathcal{C} = \mathrm{Ind}(E)$ be a compactly generated stable presentable ∞ -category with a t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. The full subcategory $\mathcal{D}(\mathbb{A}^1, \mathcal{C})_{\geq 0} \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})$ of endomorphisms (x, e) where $x \in \mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ defines the non-negative part of a t -structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$, where $(y, f) \in \mathcal{D}(\mathbb{A}^1, \mathcal{C})_{\leq 0}$ if and only if $y \in \mathcal{C}_{\leq 0}$. Moreover, the truncation functors induced by the t -structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ preserve the full subcategory $\mathcal{D}(\mathbb{A}^1, \mathcal{C}) \supseteq \mathcal{D}(\mathbb{G}_m, \mathcal{C})$.*

Proof. Requirement (1) of Definition 2.1 is inherited from \mathcal{C} . Since the truncations $\tau_{\geq 0}x$ and $\tau_{\leq 0}x$ are functorial, there is a cofiber sequence

$$(\tau_{\geq 0}x, \tau_{\geq 0}(e)) \rightarrow (x, e) \rightarrow (\tau_{\leq -1}x, \tau_{\leq -1}(e)).$$

This verifies requirement (3). As for (2), note that the forgetful functor $\mathcal{D}(\mathbb{A}^1, \mathcal{C}) \rightarrow \mathcal{C}$ detects nullhomotopic maps. This means that if $x \in \mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}_{\leq -1}$, then

$$\mathrm{Map}_{\mathcal{D}(\mathbb{A}^1, \mathcal{C})}((x, e), (y, f)) \simeq 0$$

for any endomorphisms e of x and f of y . The first claim follows.

If (x, e) is an object of $\mathcal{D}(\mathbb{G}_m, \mathcal{C})$, then e is an automorphism of x , and hence $\tau_{\geq 0}(e)$ is an automorphism of $\tau_{\geq 0}x$. So, the truncation functors preserve $\mathcal{D}(\mathbb{G}_m, \mathcal{C}) \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})$. This proves the second claim. \square

Proposition 3.14. *Let $\mathcal{C} = \mathrm{Ind}(E)$ be a compactly generated stable presentable ∞ -category with the t -structure induced (in the sense of Proposition 2.12) by a bounded t -structure on E such that E^\heartsuit is noetherian. The t -structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ of the previous lemma restricts to a bounded t -structure with noetherian heart on the full subcategory $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ of compact objects.*

Proof. Let $F \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ be the full subcategory of objects x such that $\tau_{\geq n}x$ is compact for all n . It follows immediately that F is idempotent complete. Moreover, F contains all objects of the form $(x[s], s)$ for $x \in E$ since $\tau_{\geq n}(x[s], s) \simeq (\tau_{\geq n}x, \tau_{\geq n}(s))$ and since $\tau_{\geq n}x$ is in E if x is in E . Therefore, if F is stable, the inclusion $F \rightarrow \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ is an equivalence. By definition, F is closed under suspension and desuspension. Hence, by [Lur12, Lemma 1.1.3.3], it is enough to show that F is closed under taking cofibers.

Hence, given a cofiber sequence $x \rightarrow y \rightarrow c$ in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ with $x, y \in F$, we must show that $\tau_{\geq 0}c$ is compact. Let d be the cofiber of $\tau_{\geq 0}y \rightarrow \tau_{\geq 0}y$, so that d fits into a second cofiber sequence $d \rightarrow \tau_{\geq 0}c \rightarrow \pi$, where $\pi \in \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\heartsuit$ is the image of $\pi_0c \rightarrow \pi_{-1}x$. As d is compact by the hypothesis on x and y , it is enough to show that π is compact in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$.

Let $A = E^\heartsuit$, and let $A[s]$ be the full subcategory of compact objects in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\heartsuit$. Note that this is well-defined because $(\mathrm{Mod}_{\mathbb{S}[s]}^{\mathrm{cn}} \otimes \mathcal{C}_{\geq 0})^\heartsuit \simeq \mathrm{Mod}_{\mathbb{S}[s]}^\heartsuit \otimes \mathcal{C}^\heartsuit$ by definition of the tensor product of Grothendieck abelian categories in [Lur16]. In general, there is no reason for $A[s]$ to be an abelian category. However, this is implied by the fact that A is noetherian in our case, as this ensures that the kernel of a map between finitely presented objects is again finitely presented. Moreover, it is a consequence that $\pi_n x \in A[s]$ whenever $x \in \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$. Because $A[s]$ is noetherian (by an easy generalization of the Hilbert basis theorem in algebra), it is a Serre subcategory of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\heartsuit$, so that $\pi \subseteq \pi_{-1}x$ is in $A[s]$. We are reduced to proving that if π is an object of $A[s]$, then π is compact as an object of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$.

It is convenient for the rest of the proof to write s for the endomorphism of any object of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$. Let $F_i\pi = \ker(s^i : \pi \rightarrow \pi)$ for $i \geq 0$. This is an increasing filtration on π , which stabilizes at some $F_N\pi$ for $N \geq 0$ since $A[s]$ is noetherian. Each $F_i\pi/F_{i-1}\pi$ is in fact an object of A as it is a finitely presented object of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\heartsuit$ such that s acts as zero. So, inductively, $F_N\pi$ is compact. Let τ be the quotient $\pi/F_N\pi$. The endomorphism s acts injectively on τ by construction. To see that τ is compact, choose a surjection $a[s] \rightarrow \tau$ such that $a \in A$. Let σ be the kernel, and let $\sigma_i \subseteq a \cdot s^i$ be the intersection of the kernel and $a \cdot s^i \subseteq a[s]$ (viewed as an object of $\mathrm{Ind}(A)$). Then, $s : \sigma_i \rightarrow \sigma_{i+1}$ and $\sigma \cong \bigoplus_{i \geq 0} \sigma_i$. Moreover, $s : \sigma_i \rightarrow \sigma_{i+1}$ is an isomorphism for all i . The injectivity follows from the fact that s acts injectively on $a[s]$, while the surjectivity follows (via the snake lemma) from the fact that s acts injectively on τ and $s : a \cdot s^i \rightarrow a \cdot s^{i+1}$ is surjective. It follows that $\sigma \cong \sigma_0[s]$. But, since $\sigma_0 \subseteq a$, it follows that σ is compact. Therefore, τ is compact.

Now, to see that the t -structure is bounded, it is enough to see that each object $(x[s], s)$ is bounded for $x \in E$. This is the case by construction. Finally, we have already mentioned that $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^{\omega, \heartsuit} \simeq A[s]$ is noetherian. \square

Remark 3.15. Noetherianity is used in a couple primary locations in the proof. The first is to check that $\pi_{-1}x$ is finitely presented and that π is therefore itself in $A[s]$. The second is to guarantee that the filtration $F_\bullet \pi$ stabilizes. We return in the next sections to the problem of weakening the noetherian hypothesis.

Lemma 3.16. *Let $\mathcal{C} = \text{Ind}(E)$ be a compactly generated stable presentable ∞ -category with a t -structure induced (in the sense of Proposition 2.12) by a bounded t -structure on E such that E^\heartsuit is noetherian. Then, the t -structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ respects $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega$.*

Proof. Note that we did not prove in general that the t -structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$ restricts to a t -structure on $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$. But, this is true under the noetherianity condition for the compact objects. Indeed, an object x of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ is contained in the subcategory $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega$ if and only if s^N acts nullhomotopically on x for some $N \geq 0$. If s^N does act nullhomotopically on x , then it does so on $\tau_{\geq 0}x$ as well, which shows that if $x \in \mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega$, then so is $\tau_{\geq 0}x$. \square

Corollary 3.17. *Let $\mathcal{C} = \text{Ind}(E)$ be a compactly generated stable presentable ∞ -category with a t -structure induced (in the sense of Proposition 2.12) by a bounded t -structure on E such that E^\heartsuit is noetherian. Then, there is a bounded t -structure on $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ with noetherian heart.*

Proof. The subcategory

$$\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^{\omega, \heartsuit} \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})^{\omega, \heartsuit}$$

is in fact a Serre subcategory. Indeed, if $\tau \subseteq \sigma$ is a subobject where s acts nilpotently on σ , then s acts nilpotently on τ as well. Using (i) implies (ii) in Proposition 2.19, we see that there is an induced t -structure on $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ and that the functor $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega \rightarrow \mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ is t -exact. Since every object of $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ is a retract of an object in the image of the localization functor and since the t -structure on $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ is bounded, it follows that the t -structure on $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ is bounded too. The abelian category $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^{\omega, \heartsuit}$ is noetherian because it is equivalent to the localization of the noetherian abelian category $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^{\omega, \heartsuit}$ by the Serre subcategory $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^{\omega, \heartsuit}$. \square

Proof of Theorem 3.6. Let $\mathcal{C} = \text{Ind}(E)$. Applying K -theory to the exact sequence

$$\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega \rightarrow \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega \rightarrow \mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega,$$

we obtain a cofiber sequence

$$K_{\{0\}}(\mathbb{A}^1, \mathcal{C}) \rightarrow K(\mathbb{A}^1, \mathcal{C}) \rightarrow K(\mathbb{G}_m, \mathcal{C})$$

of nonconnective K -theory spectra.

Consider the exact functor $i : \mathcal{C} \rightarrow \mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})$ given by $i(x) \simeq (x, 0)$. Since $(x[s], s) \xrightarrow{s} (x[s], s) \rightarrow (x, 0)$ is a cofiber sequence in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$, we see that $(x, 0)$ is compact if x is. Hence, i restricts to a functor $E \rightarrow \mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega$, also denoted i . Moreover, the additivity theorem, applied to this same cofiber sequence, viewed as a cofiber sequence of functors $E \rightarrow \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$, induces a nullhomotopic map $K(E) \rightarrow K(\mathbb{A}^1, \mathcal{C})$. The underlying object functor $u : \mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega \rightarrow E$ sending (x, e) to x gives $u \circ i \simeq \text{id}_E$. It follows that $K(E)$ is a summand of $K_{\{0\}}(\mathbb{A}^1, \mathcal{C})$ and that this summand maps trivially to $K(\mathbb{A}^1, \mathcal{C})$.

Now, suppose that $K_{-m}(F) = 0$ for all $1 \leq m \leq n$ and all stable ∞ -categories F which admit bounded t -structures with noetherian hearts. The remarks above prove that $K_{-n-1}(E)$ is a subquotient of $K_{-n}(\mathbb{G}_m, \mathcal{C})$. By Corollary 3.17, there is a bounded t -structure on $\mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$ with noetherian heart. Hence, $K_{-n}(\mathbb{G}_m, \mathcal{C}) = 0$ by the inductive hypothesis and so $K_{-n-1}(E) = 0$ as well. \square

3.3 The nonconnective theorem of the heart

In this section we prove Conjecture C in the case of a noetherian heart.

Theorem 3.18 (Nonconnective theorem of the heart). *If E is a small stable ∞ -category with a bounded t -structure such that E^\heartsuit is noetherian, then the natural map*

$$K(E^\heartsuit) \xrightarrow{\sim} K(E)$$

is an equivalence.

To give the theorem content, we must define $K(A)$ when A is an abelian category, show that this agrees with other definitions in the literature, and define the map $K(E^\heartsuit) \rightarrow K(E)$. We will use the terminology and results about prestable ∞ -categories of [Lur16, Appendix C], which in turn follows work of Krause [Kra15] on homotopy categories of injective complexes.

Lemma 3.19. *If A is a small abelian category, then $\text{Ind}(A)$ is a Grothendieck abelian category, the Yoneda embedding $A \rightarrow \text{Ind}(A)$ is exact, and the natural map $A \rightarrow \text{Ind}(A)^\omega$ is an equivalence.*

Proof. Since A has finite colimits, $\text{Ind}(A)$ is presentable. Moreover, it is not difficult to see that $\text{Ind}(A)$ is abelian. To see that filtered colimits preserve monomorphisms, use that filtered colimits preserve finite limits. Yoneda is always right exact, and it preserves finite colimits that exist in A . This proves exactness. The last claim follows because A is idempotent complete. \square

Definition 3.20. Let $\check{\mathcal{D}}(\text{Ind}(A))$ denote the unseparated derived ∞ -category of $\text{Ind}(A)$ as defined in [Lur16, Section C.5.8]. It is the dg nerve of the dg category of complexes of injective objects in $\text{Ind}(A)$. In particular, by [Lur12, Remark 1.3.2.3], $\text{Ho}(\check{\mathcal{D}}(\text{Ind}(A)))$ is the homotopy category of injectives as studied by Krause [Kra15]. There is a right complete t -structure on $\check{\mathcal{D}}(\text{Ind}(A))$, and $\check{\mathcal{D}}(\text{Ind}(A))_{\geq 0}$ is anticomplete (see [Lur16, Section C.5.5]) with an important universal property: it is initial among Grothendieck prestable ∞ -categories \mathcal{C} with $\mathcal{C}^\heartsuit \simeq \text{Ind}(A)$ (see [Lur16, Corollary C.5.8.9]).

Definition 3.21. Let A be a small abelian category. We define the bounded derived ∞ -category of A to be $\mathcal{D}^b(A) = \check{\mathcal{D}}(\text{Ind}(A))^\omega$. In particular, $\mathcal{D}^b(A)$ is a small idempotent complete stable ∞ -category.

Lemma 3.22. *The ∞ -category $\check{\mathcal{D}}(\text{Ind}(A))$ is compactly generated by $\mathcal{D}^b(A)$.*

Proof. This is the content of [Kra15, Theorem 4.9]. In the setting of [Lur16, Appendix C], we invoke the fact that $\check{\mathcal{D}}(\text{Ind}(A))_{\geq 0}$ is coherent (by [Lur16, Corollary C.6.5.9]) and anticomplete to conclude that $\check{\mathcal{D}}(\text{Ind}(A))_{\geq 0}$ is compactly generated by [Lur16, Theorem 6.7.1]. Since $\check{\mathcal{D}}(\text{Ind}(A))$ is right complete, a compact object of $\check{\mathcal{D}}(\text{Ind}(A))_{\geq 0}$ is compact when viewed in $\check{\mathcal{D}}(\text{Ind}(A)) \simeq \text{Sp}(\check{\mathcal{D}}(\text{Ind}(A))_{\geq 0})$. Let $y \in \check{\mathcal{D}}(\text{Ind}(A))$. We have to show that if $\text{Map}_{\check{\mathcal{D}}(\text{Ind}(A))}(x, y)$ for all x in $\check{\mathcal{D}}(\text{Ind}(A))_{\geq n}^\omega$ and all n , then $y \simeq 0$. But, if this condition is satisfied, then $y \in \check{\mathcal{D}}(\text{Ind}(A))_{\leq n}$ for all n . Since $\check{\mathcal{D}}(\text{Ind}(A))$ is right separated, $y \simeq 0$. \square

Lemma 3.23. *The canonical t -structure on $\check{\mathcal{D}}(\text{Ind}(A))$ restricts to a bounded t -structure on $\mathcal{D}^b(A)$ with heart equivalent to A .*

Proof. This follows from [Lur16, Theorem C.6.7.1]. \square

Lemma 3.24. *Let $\mathcal{D}' \subseteq \mathcal{D}(\text{Ind}(A))$ denote the full subcategory of objects x such that $H_n(x) \in A \subseteq \text{Ind}(A)$ for all n and such that $H_n(x)$ is non-zero for at most finitely many $n \in \mathbb{Z}$. Then, the map $\check{\mathcal{D}}(\text{Ind}(A)) \rightarrow \mathcal{D}(\text{Ind}(A))$ restricts to an equivalence $\mathcal{D}^b(A) \rightarrow \mathcal{D}'$.*

Proof. The claim can be checked at the level of homotopy categories, which is the other part of [Kra15, Theorem 4.9]. \square

Definition 3.25. If A is a small abelian category, then we define $K(A) = K(\mathcal{D}^b(A))$, the nonconnective K -theory spectrum of the idempotent complete stable ∞ -category $\mathcal{D}^b(A)$ (defined in [BGT13]).

We want to construct “the natural map $K(E^\heartsuit) \rightarrow K(E)$ ” of the statement of Theorem 3.18.

Proposition 3.26. *Let E be a small stable ∞ -category with a bounded t -structure. Then, there is a natural t -exact functor $\mathcal{D}^b(E^\heartsuit) \rightarrow E$ inducing an equivalence on hearts.*

Proof. To define the natural map in the statement of Theorem 3.18, let E be a small stable ∞ -category with a bounded t -structure, and let $A = E^\heartsuit$. By [Lur16, Corollary C.5.8.9], there is a left exact functor $\check{\mathcal{D}}(\mathrm{Ind}(A))_{\geq 0} \rightarrow \mathrm{Ind}(E)_{\geq 0}$ inducing the equivalence $\mathrm{Ind}(A) \simeq \mathrm{Ind}(E)^\heartsuit$.

By [Lur16, Proposition C.3.2.1], the induced functor $F : \check{\mathcal{D}}(\mathrm{Ind}(A)) \rightarrow \mathrm{Ind}(E)$ is t -exact and induces an equivalence on hearts. It suffices to check that F preserves compact objects. By Lemma 3.24, every compact object of $\check{\mathcal{D}}(\mathrm{Ind}(E))$ is a finite iterated fiber of maps between shifts of objects in $A \subseteq \check{\mathcal{D}}(\mathrm{Ind}(E))^\heartsuit$. Thus, it suffices to show that $F(x) \in E$ when $x \in A$. But, this follows from hypothesis. \square

Corollary 3.27. *Let E be a small stable ∞ -category with a bounded t -structure. Then, there is a natural map $K(E^\heartsuit) \rightarrow K(E)$ of nonconnective K -theory spectra.*

Proof. Apply K to the exact functor $\mathcal{D}^b(E^\heartsuit) \rightarrow E$. \square

With this in mind, we turn to the proof of the theorem.

Proof of Theorem 3.18. The first step is to use Barwick’s theorem of the heart to prove that the induced map $K^{\mathrm{cn}}(E^\heartsuit) \rightarrow K^{\mathrm{cn}}(E)$ of connective K -theory is an equivalence. Philosophically, this is Barwick’s theorem, but we have defined K -theory in terms of stable ∞ -categories instead of using exact ∞ -categories.

Consider the commutative triangle

$$\begin{array}{ccc} & E^\heartsuit & \\ \swarrow & & \searrow \\ \mathcal{D}^b(E^\heartsuit) & \xrightarrow{\quad} & E \end{array}$$

of Waldhausen ∞ -categories in the sense of [Bar15]. For the moment, denote by K^{Bar} the (connective) K -theory spectrum of a Waldhausen ∞ -category, as constructed in [Bar16]. Then, [Bar15, Theorem 6.1] shows that the $K^{\mathrm{Bar}}(E^\heartsuit) \rightarrow K^{\mathrm{Bar}}(\mathcal{D}^b(E^\heartsuit))$ and $K^{\mathrm{Bar}}(E^\heartsuit) \rightarrow K^{\mathrm{Bar}}(E)$ are equivalences. Hence, $K^{\mathrm{Bar}}(E^\heartsuit) \rightarrow K^{\mathrm{Bar}}(E)$ is an equivalence.

By [Bar16, Corollary 10.6], $K^{\mathrm{Bar}}(\mathcal{D}^b(E^\heartsuit))$ and $K^{\mathrm{Bar}}(E)$ are equivalent to the Waldhausen K -theory of suitable Waldhausen categories, and these are in turn equivalent to $K^{\mathrm{cn}}(\mathcal{D}^b(E^\heartsuit))$ and $K^{\mathrm{cn}}(E)$, respectively, by [BGT13, Theorem 7.8]. This proves the result in connective K -theory.

Now, in the situation of the theorem, both $\mathcal{D}^b(E^\heartsuit)$ and E have bounded t -structures with noetherian hearts. It follows from Theorem 3.6 that $K_{-n}(E^\heartsuit) = K_{-n}(E) = 0$ for $n \geq 1$. This completes the proof. \square

Remark 3.28. If E is a small stable ∞ -category with a bounded t -structure, then $K^{\mathrm{cn}}(E) \simeq K^{\mathrm{cn}}(\mathcal{D}^b(E^\heartsuit))$ is equivalent to the Quillen K -theory of E^\heartsuit viewed as an exact category. This follows from the Gillet-Waldhausen theorem [TT90, Theorem 1.11.7] and a theorem of Waldhausen [TT90, Theorem 1.11.2]. In the end, the theorem of the heart is a generalization of the Gillet-Waldhausen theorem.

Remark 3.29. Note that the negative K -groups of a small abelian category \mathcal{A} , defined in this paper as $K_{-n}(\mathcal{D}^b(\mathcal{A}))$ for $n \geq 1$, agree with the negative K -groups of \mathcal{A} as defined by Schlichting [Sch06]. To see this, denote the latter by $K_{-n}^S(\mathcal{A})$ for the moment.

As it is not necessary for our paper, we only illustrate the argument. Recall that a Frobenius pair is a pair $(\mathcal{E}, \mathcal{E}_0)$ of small Frobenius categories where \mathcal{E}_0 is a full subcategory of \mathcal{E} such that the embedding $\mathcal{E}_0 \rightarrow \mathcal{E}$ preserves projective (or equivalently injective) objects. A morphism of Frobenius pairs $(\mathcal{E}, \mathcal{E}_0) \rightarrow (\mathcal{F}, \mathcal{F}_0)$ consists of a functor $\Phi : \mathcal{E} \rightarrow \mathcal{F}$ such that $\Phi(M) \in \mathcal{F}_0$ if $M \in \mathcal{E}_0$. (See for example [Sch06].) Let $\mathcal{F}\text{rob}$ denote the ∞ -category of Frobenius pairs. Using the functor $\mathcal{D}_{\text{sing}}$ defined in Appendix A, we obtain a functor $\mathcal{D}_{\text{sing}} : \mathcal{F}\text{rob} \rightarrow \text{Cat}_{\infty}^{\text{perf}}$ defined by letting

$$\mathcal{D}_{\text{sing}}(\mathcal{E}, \mathcal{E}_0) = (\mathcal{D}_{\text{sing}}(\mathcal{E})/\mathcal{D}_{\text{sing}}(\mathcal{E}_0))^{\sim},$$

the idempotent completion of the Verdier quotient. Note that $\mathcal{D}_{\text{sing}}(\mathcal{E}_0) \rightarrow \mathcal{D}_{\text{sing}}(\mathcal{E})$ is fully faithful because any map \mathcal{E}_0 that factors through a projective in \mathcal{E} also factors through a projective in \mathcal{E}_0 .

By the definition of an exact sequence in $\mathcal{F}\text{rob}$ given in [Sch06], $\mathcal{D}_{\text{sing}}$ sends exact sequences to localization sequences. Moreover, if $(\mathcal{E}, \mathcal{E}_0)$ is a flasque Frobenius pair, meaning that there is an endofunctor T of the pair such that $T \simeq T \oplus \text{id}$, then $\mathcal{D}_{\text{sing}}(\mathcal{E}, \mathcal{E}_0)$ is flasque. Using these facts, and the fact that K_0 can be computed either in $\mathcal{F}\text{rob}$ or in $\text{Cat}_{\infty}^{\text{perf}}$, it follows that $K_{-n}^S(\mathcal{A}) \cong K_{-n}(A)$.

3.4 Counterexamples using non-stably coherent rings

While our proof of Theorem 3.6 uses crucially the hypothesis that the small stable ∞ -category E has a bounded t -structure with noetherian heart, much of the proof works for a general E with any bounded t -structure. In particular, the existence of the sequence $K_{\{0\}}(\mathbb{A}^1, \mathcal{C}) \rightarrow K(\mathbb{A}^1, \mathcal{C}) \rightarrow K(G_m, \mathcal{C})$, where $\mathcal{C} = \text{Ind}(E)$, exists without any hypothesis on E except that it be small and stable, as does the fact that $K(E)$ itself is a summand of $K_{\{0\}}(\mathbb{A}^1, \mathcal{C})$. The fact that $K_{-1}(E) = 0$ whenever E admits a bounded t -structure provides additional strength to the assertion that $K_{-n}(E)$ should be zero for all $n \geq 1$.

The noetherian hypothesis is used to prove that the t -structure of Lemma 3.13 on $\mathcal{D}(G_m, \mathcal{C})$ restricts to a (bounded) t -structure on $\mathcal{D}(G_m, \mathcal{C})^{\omega}$. This leads to the inductive step. One may ask if this is true in general, with a different proof. This is not the case.

Let R be an ordinary ring. A finitely presented right R -module M is coherent if every finitely generated submodule $N \subseteq M$ is finitely presented. The ring R is **right coherent** if R is coherent as a right R -module. We say that R is **right regular** if every finitely presented right R -module has finite projective dimension. Finally, we call R **right regular coherent** if it is both right coherent and right regular.

Theorem 3.30. *If R is a right regular coherent ring, then $K_{-1}(R) = 0$.*

Proof. The right coherence of R means that the category $\text{Mod}_R^{\heartsuit, \omega}$ of finitely presented R -modules is abelian and hence that $\mathcal{D}^b(R) = \mathcal{D}^b(\text{Mod}_R^{\heartsuit, \omega})$ is a well-defined small stable presentable ∞ -category. The right regularity of R means that the natural map $\text{Mod}_R^{\omega} \rightarrow \mathcal{D}^b(R)$ is an equivalence. Hence, $K(R) \simeq K(\mathcal{D}^b(R))$. We can conclude in either of two ways. We can appeal to Theorem 2.34 using that the equivalence induces a bounded t -structure on Mod_R^{ω} . Or, we can use that $K_{-1}(\mathcal{D}^b(R)) = 0$, as proved by Schlichting [Sch06, Theorem 6]. \square

Note that if R is an ordinary ring, then $\mathcal{D}(G_m, \text{Mod}_R) \simeq \text{Mod}_{R[s^{\pm 1}]}$ and the t -structure on $\mathcal{D}(G_m, \text{Mod}_R)$ induced by that on Mod_R via Lemma 3.13 agrees with the standard t -structure on $\text{Mod}_{R[s^{\pm 1}]}$.

Proposition 3.31. *Suppose that R is right regular coherent but that $R[s^{\pm 1}]$ is not right coherent. Then, the t -structure on $\text{Mod}_{R[s^{\pm 1}]}$ does not restrict to a t -structure on compact objects.*

Proof. Note that $R[s^{\pm 1}]$ is right regular. Let $I \subseteq R[s^{\pm 1}]$ be a finitely generated ideal that is not finitely presented, and let $P : R[s^{\pm 1}]^n \rightarrow R[s^{\pm 1}]$ be a chain complex in degrees 1 and 0, with $d(R[s^{\pm 1}]^n) = I$. By our choice of I , $H_1(P)$ is not a finitely generated $R[s^{\pm 1}]$ -module. If the t -structure on $\text{Mod}_{R[s^{\pm 1}]}$ restricted to a t -structure on the compact objects, then $H_1(P)$ would have to be perfect, and hence finitely presented, a contradiction. \square

Example 3.32. Glaz presents an example of Soublin in [Gla89, Example 7.3.13] showing that the ring

$$R = \prod_{m \in \mathbb{Z}} \mathbb{Q}[[x, y]]$$

is right regular coherent, but that $R[s]$ is not coherent. By [Gla89, Theorem 8.2.4(2)], $R[s^{\pm 1}]$ is not coherent either.

This example implies that the strategy used in the previous section to prove Conjecture B in the case of a noetherian heart cannot work for general small stable ∞ -categories E equipped with bounded t -structures.

Another strategy is to replace $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega$ by some stable ∞ -category that does have a bounded t -structure. For example, one can consider the abelian closure $\widetilde{\text{coh}}(\mathbb{A}^1, \mathcal{C})$ in $\mathcal{D}(\mathbb{A}^1, \mathcal{C})^\heartsuit$ of the additive category consisting of $\pi_n x$ as x ranges over all compact objects of $\mathcal{D}(\mathbb{A}^1, \mathcal{C})$. Let $\mathcal{D}_{\text{coh}}^b(\mathbb{A}^1, \mathcal{C}) \subseteq \mathcal{D}(\mathbb{A}^1, \mathcal{C})$ be the full subcategory of bounded objects x such that $\pi_n x \in \widetilde{\text{coh}}(\mathbb{A}^1, \mathcal{C})$ for all n . This is a small stable ∞ -category and the induced t -structure is bounded.

Now, consider the localization sequence

$$\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C}) \cap \mathcal{D}_{\text{coh}}^b(\mathbb{A}^1, \mathcal{C}) \rightarrow \mathcal{D}_{\text{coh}}^b(\mathbb{A}^1, \mathcal{C}) \rightarrow \left(\mathcal{D}_{\text{coh}}^b(\mathbb{A}^1, \mathcal{C}) / \mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C}) \cap \mathcal{D}_{\text{coh}}^b(\mathbb{A}^1, \mathcal{C}) \right)^\sim.$$

Let $\mathcal{D}_{\{0\}, \text{coh}}^b(\mathbb{A}^1, \mathcal{C})$ denote the left-hand side. For this to play the role of the localization sequence $\mathcal{D}_{\{0\}}(\mathbb{A}^1, \mathcal{C})^\omega \rightarrow \mathcal{D}(\mathbb{A}^1, \mathcal{C})^\omega \rightarrow \mathcal{D}(\mathbb{G}_m, \mathcal{C})^\omega$, we need to guarantee two things:

(1) $K(E)$ is a summand of $K(\mathcal{D}_{\{0\}, \text{coh}}^b(\mathbb{A}^1, \mathcal{C}))$;

(2) the abelian subcategory

$$\mathcal{D}_{\{0\}, \text{coh}}^b(\mathbb{A}^1, \mathcal{C})^\heartsuit \subseteq \mathcal{D}_{\text{coh}}^b(\mathbb{A}^1, \mathcal{C})^\heartsuit$$

is Serre.

Condition (2) is easier and is in fact always true. Condition (1) would follow if x is compact when (x, e) is an object of $\mathcal{D}_{\{0\}, \text{coh}}^b(\mathbb{A}^1, \mathcal{C})$.

3.5 Stable coherence

The next theorem was known to Bass and Gersten, but our proof is slightly simpler.

Theorem 3.33. *Suppose that R is a right regular coherent ring such that $R[s_1, \dots, s_n]$ is right coherent. Then, $K_{-n-1}(R) = 0$.*

Proof. It is enough to note that in this case $R[s_1^{\pm 1}, \dots, s_n^{\pm 1}]$ is right coherent by [Gla89, Theorem 8.2.4(2)]. Hence, $K_{-1}(R[s_1^{\pm 1}, \dots, s_n^{\pm 1}]) = 0$. As $K_{-n-1}(R)$ is a subquotient of $K_{-1}(R[s_1^{\pm 1}, \dots, s_n^{\pm 1}])$, the result follows from Theorem 3.30. \square

The classical proof (due to Bass [Bas73, Section 2]) uses a specific inductive presentation of $K_{-n-1}(R)$, namely as the cokernel of

$$K_{-n}(R[s]) \oplus K_{-n}(R[s^{-1}]) \rightarrow K_{-n}(R[s^{\pm 1}]).$$

(See [TT90, Section 6].) In particular, $K_{-n-1}(R)$ is a quotient of $K_{-1}(R[s_1^{\pm 1}, \dots, s_n^{\pm 1}])$, which vanishes if $R[s_1^{\pm 1}, \dots, s_n^{\pm 1}]$ is right regular coherent.

One reason to prefer our proof is that it extends immediately to small abelian categories A such that $A[s_1, \dots, s_n]$ is abelian, with notation as in the proof of Proposition 3.14. We know of no analogous general result using Bass' methods in the literature. We restate this result separately.

Theorem 3.34. *Let A be a small abelian category such that $A[s_1, \dots, s_n]$ is abelian. Then, $K_{-n-1}(A) = 0$.*

It seems that Schlichting's paper is very close to establishing a result like this. However, the proof given of [Sch06, Lemma 8] relies on the structure on injective modules in a noetherian abelian category to establish the long exact sequence in K -groups allowing one to conclude that $K_{-n-1}(A)$ is a subquotient of $K_{-1}(A[s_1^{\pm 1}, \dots, s_n^{\pm 1}])$. So, that paper allows one to prove the theorem for A noetherian ([Sch06, Remark 7]), but not in this stably coherent setting.

We close this section with a discussion relating these vanishing results and the K -theoretic Farrell-Jones conjecture. None of these results are new (as they all follow from Theorem 3.33), but it serves to illustrate the importance of Conjectures A, B, and C in the non-noetherian case.

The most interesting cases of the conjecture are when $R = \mathbb{Z}$ or when R is an arbitrary regular noetherian commutative ring. Farrell and Jones [FJ95] proved that $K_{-n}(\mathbb{Z}[V]) = 0$ for $n \geq 2$. If the Farrell-Jones conjecture holds for G , then it follows from the homotopy colimit spectral sequence that $K_{-n}(\mathbb{Z}[G]) = 0$ for $n \geq 2$ as well. In many cases it is suspected that $K_{-n}(R[G]) = 0$ for all $n \geq 1$. For example, this follows from the Farrell-Jones conjecture when the orders of all finite subgroups of G are invertible in R (see [LR05, Conjecture 79]). Our application to this problem is via a class of groups studied in this setting by Waldhausen [Wal78]. Say that a group G is **regular coherent** (resp. **noetherian**) if $R[G]$ is right regular coherent (resp. **noetherian**) for any regular noetherian commutative ring R .

Theorem 3.35. *Let R be a regular noetherian commutative ring, and let G be a regular coherent group. Then, $K_{-n}(R[G]) = 0$ for $n \geq 1$.*

Proof. The key point is that $R[G][s] \cong (R[s])[G]$, so that under the hypotheses, $R[G]$ is stably coherent. The result follows from Theorem 3.33. \square

The Farrell-Jones conjecture is known in many cases, and hence the vanishing of negative K -theory is known in many cases over the integers by the result of Farrell and Jones. For a table of known results on the Farrell-Jones conjecture, see [LR05, Section 2.6.3].

Example 3.36. Many groups are regular and coherent. The following list is transcribed from [Wal78]. The group G is regular coherent if it is (1) a free group, (2) a free abelian group, (3) a polycyclic group, (4) a torsion-free one-relator group, (5) a group of the form $\pi_1 M$ where M is a 2-manifold not homeomorphic to $\mathbb{R}P^2$, (6) a sufficiently large 3-manifold group, (7) a group of the form $\pi_1 M$ where M is a submanifold of S^3 , (8) a subgroup of a group of one of the above types, or (9) a filtered colimit of inclusions thereof. In particular, for all of these groups, $K_{-n}(R[G]) = 0$ for $n \geq 1$. Example (8) is particularly interesting as regular coherence passes to subgroups even though this is not known for the Farrell-Jones conjecture.

3.6 Serre cones of abelian categories

We have seen that the straightforward generalization of Schlichting's inductive strategy to prove vanishing of negative K -theory of noetherian abelian categories founders because of the failure of the Serre subcategory condition on the hearts, even though the weak Serre subcategory condition always holds.

Part of the subtlety of Schlichting's conjecture is that the negative K -theory of an abelian category is defined using derived categories. To date, there is no definition internal to abelian categories. Given a small idempotent complete stable ∞ -category E and an uncountable regular cardinal κ , let $\Sigma_\kappa(E)$ be the cofiber in $\text{Cat}_\infty^{\text{perf}}$ fitting into the exact sequence

$$E \rightarrow \text{Ind}(E)^\kappa \rightarrow \Sigma_\kappa(E).$$

This allows us to define negative K -theory inductively as $K_{-n}(E) = K_0(\Sigma_\kappa^{(n)}(E))$ since $K(\text{Ind}(E)^\kappa) \simeq 0$.

No similar construction is known for the negative K -theory of abelian categories. It would be possible given a positive answer to the next question.

Question 3.37. Suppose that A is a small abelian category. Does there exist an exact fully faithful inclusion of abelian categories $A \subseteq B$ such that $K(B) \simeq 0$ and such that A is Serre in B ?

For example, B might be closed under countable coproducts, which implies the K -acyclicity condition. One natural guess would be to take a category $\text{Ind}(A)^\kappa$ of κ -compact objects for an uncountable cardinal κ . However, $A \subseteq \text{Ind}(A)^\kappa$ is not typically Serre.

Example 3.38. Let R be a non-noetherian coherent ring, and let $\text{coh}_R \subseteq \text{Mod}_R^{\heartsuit, \kappa}$ be the full subcategory of coherent right R -modules inside all κ -compact R -modules (for some regular uncountable cardinal κ). Then, R itself has subobjects (specifically, non-finitely generated ideals) in $\text{Mod}_R^{\heartsuit, \kappa}$ not contained in coh_R .

Proposition 3.39. Conjecture C and a positive answer to Question 3.37 imply Conjectures A and B.

Proof. For $n \geq 1$, let $A(n)$ denote the statement that $K_{-n}(A) = 0$ for all small abelian categories A , and let $B(n)$ denote the statement that $K_{-n}(E) = 0$ for every small stable ∞ -category E with a bounded t -structure. Since we are assuming Conjecture C, $A(n)$ if and only if $B(n)$. So, assume $B(n)$ for some $n \geq 1$. It suffices to prove $A(n+1)$.

Let A be a small abelian category, and let $A \subseteq B$ denote the abelian category guaranteed by the hypothesis of the proposition. Conjecture C implies that $K(A) = K(\mathcal{D}^b(A)) \simeq K(\mathcal{D}_A^b(B))$. (Note that even this special case of Conjecture C is open: see [Wei13, Open Problem V.5.3].) Consider the localization sequence

$$\mathcal{D}_A^b(B) \rightarrow \mathcal{D}^b(B) \rightarrow (\mathcal{D}^b(B)/\mathcal{D}_A^b(B))^\sim.$$

By our choice of B , $K(\mathcal{D}^b(B)) \simeq 0$, so

$$K_{-n}((\mathcal{D}^b(B)/\mathcal{D}_A^b(B))^\sim) \rightarrow K_{-n-1}(\mathcal{D}_A^b(B)) \cong K_{-n-1}(A)$$

is surjective. The quotient $(\mathcal{D}^b(B)/\mathcal{D}_A^b(B))^\sim$ has a bounded t -structure by Proposition 2.19. Thus, by $B(n)$, $K_{-n-1}(A) = 0$. \square

Remark 3.40. As a final philosophical remark, note that negative K -theory exists because of the need to idempotent complete when constructing a localization of stable ∞ -categories. Since abelian categories are idempotent complete, the sequences $A \rightarrow B \rightarrow B/A$ are already exact when $A \subseteq B$ is Serre. In particular, the induced map $K_0(B) \rightarrow K_0(B/A)$ is always surjective for such a localization sequence. It follows that, to the extent it exists along the lines of [BGT13], the universal localizing invariant of abelian categories should actually be connective K -theory. This does not imply Schlichting's conjecture by itself, but it would provide some evidence.

4 Negative K -theory of dg algebras and ring spectra

We present here applications of the vanishing results above to the K -theory of dg algebras and of ring spectra.

4.1 Negative K -theory of dg algebras

Corollary 1.4 follows easily from our results.

Theorem 4.1. *Let k be a commutative ring, and let A be a cohomological dg k -algebra such that $H^0(A)$ is semisimple, $H^i(A)$ is a finitely generated right $H^0(A)$ -module for all i , and $H^i(A) = 0$ for $i < 0$. Then, $K_{-n}(A) = 0$ for $n \geq 1$.*

Proof. Keller and Nicolás prove in [KN13, Theorem 7.1] that under these hypotheses, Mod_A^ω admits a bounded t -structure whose heart is a length category. Recall that a length category is a small abelian category in which every object has finite length. In particular, it is noetherian. The result follows now from Theorem 3.18. \square

Example 4.2. Suppose that k is a field and R is a noetherian local commutative k -algebra with maximal ideal \mathfrak{m} . Then, the derived endomorphism algebra A of R/\mathfrak{m} , which computes $\text{Ext}_R^*(R/\mathfrak{m}, R/\mathfrak{m})$ satisfies the hypotheses of the theorem, and hence the negative K -theory of A vanishes. We can see this in another way. Let $\mathcal{A} \subseteq \text{Mod}_R^{\vee, \omega}$ be the full subcategory of finitely presented R -modules supported set theoretically on $\text{Spec } R/\mathfrak{m} \subseteq \text{Spec } R$. Then, $\mathcal{D}^b(\mathcal{A})$ is a fully subcategory of $\mathcal{D}^b(R)$ and R/\mathfrak{m} is a compact generator. Hence, $\text{Mod}_A^\omega \simeq \mathcal{D}^b(\mathcal{A})$. So, since R is noetherian, \mathcal{A} is noetherian, and the fact that $K_{-n}(A) = 0$ for $n \geq 1$ follows from Schlichting's theorem.

Example 4.3. If k is a field and X is a smooth proper k -scheme, then the algebraic de Rham complex, which computes the algebraic de Rham cohomology $H_{\text{dR}}^*(X/k)$, satisfies the conditions of the theorem.

4.2 Negative K -theory of periodic and related ring spectra

Let R be a connective ring spectrum. A right R -module M is π_* -**finitely presented** if $\bigoplus_n \pi_n M$ is a finitely presented (right) $\pi_0 R$ -module. In particular, this means that M is bounded and that each $\pi_n M$ is a finitely presented $\pi_0 R$ -module.² A discrete ring R is said to be **right noetherian** if every submodule of a finitely generated R -module is finitely generated. A connective ring spectrum R is **right noetherian** if $\pi_0 R$ is right noetherian and if $\pi_n R$ is finitely generated as a right $\pi_0 R$ -module for all $n \in \mathbb{N}$.

Following [MR01], a discrete ring R is said to be **right regular** if every finitely generated discrete (right) R -module has finite projective dimension. A connective ring spectrum R will be said to be **right regular** if $\pi_0 R$ is right regular and if each π_* -finitely presented (right) R -module spectrum M is compact. A connective ring spectrum R will be called **right regular noetherian** if it is right noetherian and right regular.

For the purposes of this section, a map $R \rightarrow S$ of ring spectra will be called a **localization** if the induced map $\text{Mod}_R \rightarrow \text{Mod}_S$ is a localization with kernel generated by a compact object, or equivalently by a finite set of compact objects.

The next result extends those of Barwick and Lawson in [BL14].

Proposition 4.4. *Let R be a right regular noetherian ring spectrum. Suppose that $R \rightarrow S$ is a localization of R such that for a compact R -module M , $S \otimes_R M \simeq 0$ if and only if M is π_* -finitely presented. Then, there is a fiber sequence*

$$K(\pi_0 R) \rightarrow K(R) \rightarrow K(S)$$

of nonconnective K -theory spectra.

²In [BL14], π_* -finitely presented R -modules are called *coherent*. We prefer our terminology because a right coherent ring spectrum R in the sense of [Lur12, Definition 7.2.5.16] need not be π_* -finitely presented as a right R -module spectrum.

Proof. Let $\text{Mod}_R^{\pi_*\text{-fp}} \subseteq \text{Mod}_R^\omega$ be the full subcategory of π_* -finitely presented R -modules. The localization theorem in algebraic K -theory gives a fiber sequence

$$K(\text{Mod}_R^{\pi_*\text{-fp}}) \rightarrow K(R) \rightarrow K(S).$$

Since R is connective, there is a bounded t -structure on $\text{Mod}_R^{\pi_*\text{-fp}}$ with noetherian heart (the category of finitely presented discrete right R -modules). The result follows from Theorem 3.18. \square

Corollary 4.5. *If R is a right regular noetherian ring spectrum and $R \rightarrow S$ is a localization of R such that for a compact R -module M , $S \otimes_R M \simeq 0$ if and only if M is π_* -finitely presented, then $K_{-n}(S) = 0$ for all $n \geq 1$.*

Proof. Indeed, $K_{-n}(\pi_0 R) = 0$ for $n \geq 1$ since R is right regular noetherian. Moreover, $K_{-n}(R) \cong K_{-n}(\pi_0 R)$ for $n \geq 1$ by [BGT13, Theorem 9.53]. \square

There are many examples of regular ring spectra admitting localizations satisfying the condition of the theorem. The consequences for negative K -theory are new and require the methods of this paper.

- Example 4.6.**
1. If R is a ring spectrum with $\pi_* R \cong \pi_0 R[u]$ where $|u| = 2m > 0$ and $\pi_0 R$ is right regular noetherian, then $R \rightarrow R[u^{-1}]$ satisfies the conditions of the theorem. In particular, if S is an even periodic ring spectrum with $\pi_0 S$ right regular noetherian, then $K_{-n}(S) = 0$ for $n \geq 1$. (Corollary 1.7.)
 2. In particular, $K_{-n}(\text{KU}) = 0$ for $n \geq 1$. This extends the theorem of Blumberg and Mandell [BM08]. (Corollaries 1.5 and 1.6.)
 3. Similarly, $K_{-n}(E_m) = 0$, $K_{-n}(K_m) = 0$, and $K_{-n}(K(m)) = 0$ for $n \geq 1$ and $m \geq 0$, where E_m is the Morava E -theory spectrum, K_m is the 2-periodic Morava K -theory spectrum, and $K(m)$ is the $2(p^m - 1)$ -periodic Morava K -theory spectrum.
 4. Barwick and Lawson show in [BL14] that ko is right regular noetherian, and that $\text{ko} \rightarrow \text{KO}$ satisfies the hypothesis of the theorem. Hence, $K_{-n}(\text{KO}) = 0$ for $n \geq 1$. (Corollaries 1.8 and 1.9.)
 5. They also show that tmf is right regular noetherian, and that $\text{tmf} \rightarrow \text{Tmf}$ satisfies the hypothesis of Proposition 4.4. Therefore, $K_{-n}(\text{Tmf}) = 0$ for $n \geq 1$. (Corollaries 1.8 and 1.9.)

Example 4.7. Not all periodic ring spectra concentrated in even degrees satisfy the hypotheses of Example 4.6(1). For example, the Johnson-Wilson theories $E(m)$ with $m \geq 2$ have

$$\pi_* E(m) = \mathbb{Z}_{(p)}[v_1, \dots, v_{m-1}, v_m^{\pm 1}],$$

where $|v_i| = 2p^i - 2$. Hence, they are periodic with period $2(p^m - 1)$, but they are not concentrated in multiples of this degree. We do not know if $K_{-n}(E(m)) = 0$ for $m \geq 2$ and $n \geq 1$.

4.3 Negative K -theory of cochain algebras

In a different direction, we consider cochain algebras, proving Corollary 1.10.

Theorem 4.8. *Let X be a compact space and R a regular noetherian discrete commutative ring. There is an equivalence $\bigoplus_{x \in \pi_0 X} K(R) \simeq K(C^*(X, R))$ of nonconnective K -theory spectra. In particular, $K_{-n}(C^*(X, R)) = 0$ for $n \geq 1$.*

Proof. It is enough to consider the case when X is connected, so that $X \simeq B\Omega X$. Let

$$\mathrm{Loc}_X(\mathrm{Mod}_R) \simeq \mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Mod}_R) \simeq \mathrm{Mod}_{C_*(\Omega X, R)}$$

be the ∞ -category of local systems on X with coefficients in the stable ∞ -category Mod_R of complexes of R -modules. Since the endomorphism algebra of the constant local system on R is $C^*(X, R)$, there is a fully faithful functor

$$\mathrm{Mod}_{C^*(X, R)} \rightarrow \mathrm{Mod}_{C_*(\Omega X, R)}.$$

As R is connective, so is $C_*(\Omega X, R)$, and hence there is an induced t -structure on $\mathrm{Loc}_X(\mathrm{Mod}_R)$.

If X is compact (in the ∞ -category of spaces), then R is compact when viewed as a $C_*(\Omega X, R)$ -module (for example by [DGI06, Proposition 5.3]). But, R corresponds to $C^*(X, R)$ under the functor above. It follows that $\mathrm{Mod}_{C^*(X, R)} \rightarrow \mathrm{Loc}_X(\mathrm{Mod}_R)$ sends compact objects to bounded objects with respect to the t -structure on $\mathrm{Loc}_X(\mathrm{Mod}_R)$. Moreover, the t -structure restricts to a t -structure on $\mathrm{Mod}_{C^*(X, R)}$ by Mathew's description [Mat16, Proposition 7.8] of the essential image as the ind-unipotent modules over $C_*(\Omega X, R)$, a condition which depends only on the action of $\pi_1 X$ on the homotopy groups of the R -module of the underlying local system. Hence, $\mathrm{Mod}_{C^*(X, R)}^\omega$ has a bounded t -structure, with heart easily seen to be the abelian category of finitely presented R -modules.

The theorem now follows immediately from the nonconnective theorem of the heart (Theorem 3.18) and the fact that $K_{-n}(R) = 0$ for $n \geq 1$. \square

A Frobenius nerves

We examine an ∞ -categorical model of the stable category of a Frobenius category. This material is used in the main body of the paper to verify that Schlichting's definition of the negative K -theory of a small abelian category \mathcal{A} agrees with the negative K -theory of the small stable ∞ -category $\mathcal{D}^b(\mathcal{A})$, as defined in [BGT13].

Let \mathcal{E} be a small exact category in the sense of Quillen [Qui73]. We will identify \mathcal{E} with a full subcategory of $\mathcal{A} = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Mod}_{\mathbb{Z}}^\heartsuit)$, the category of left exact additive functors $\mathcal{E}^{\mathrm{op}} \rightarrow \mathrm{Mod}_{\mathbb{Z}}^\heartsuit$. The (Yoneda) embedding $\mathcal{E} \rightarrow \mathcal{A}$ is exact and reflects exactness. Moreover, \mathcal{E} is closed under extensions in \mathcal{A} . If, additionally, \mathcal{E} is idempotent complete, then \mathcal{E} is closed under taking kernels of epimorphisms in \mathcal{A} . See [TT90, Proposition A.7.16]. In other words, \mathcal{E} satisfies hypothesis [TT90, 1.11.3.1], the key assumption needed for the Gillet-Waldhausen theorem [TT90, Theorem 1.11.7]. We refer to [TT90, Appendix A] in general for details about the Gabriel-Quillen embedding.

Mimicking the definitions in an abelian category, we say that an object P of \mathcal{E} is **projective** if for every admissible epi $M \twoheadrightarrow N$ the induced map $\mathrm{Hom}_{\mathcal{E}}(P, M) \rightarrow \mathrm{Hom}_{\mathcal{E}}(P, N)$ is surjective. Dually, an object I of \mathcal{E} is **injective** if $\mathrm{Hom}_{\mathcal{E}}(N, I) \rightarrow \mathrm{Hom}_{\mathcal{E}}(M, I)$ is surjective for every admissible mono $M \hookrightarrow N$ in \mathcal{E} .

We say that \mathcal{E} has **enough projectives** if for every object M of \mathcal{E} there is an admissible epi $P \twoheadrightarrow M$ where P is projective. Let $\mathcal{E}^{\mathrm{proj}}$ denote the full subcategory of projective objects of \mathcal{E} . Similarly, \mathcal{E} has **enough injectives** if for every object M of \mathcal{E} there is an admissible mono $M \hookrightarrow I$ where I is injective.

A **Frobenius category** is an exact category which has enough injectives and projectives and an object of \mathcal{E} is projective if and only if it is injective.

Construction A.1. If \mathcal{E} is a Frobenius category, the **stable category** $\underline{\mathcal{E}}$ of \mathcal{E} has the same objects as \mathcal{E} with morphisms $\mathrm{Hom}_{\underline{\mathcal{E}}}(M, N)$ the quotient of $\mathrm{Hom}_{\mathcal{E}}(M, N)$ by the subgroup of morphisms $f : M \rightarrow N$ factoring through a projective (or equivalently injective) object of \mathcal{E} .

Remark A.2. The stable category $\underline{\mathcal{E}}$ of a Frobenius category \mathcal{E} is triangulated. This was first observed by Happel [Hap87, Theorem 9.4] following ideas of A. Heller [Hel60]. The loop space of an object M is obtained by taking an admissible exact sequence $\Omega M \hookrightarrow P \twoheadrightarrow M$ with P projective. Then, ΩM is isomorphic to

$M[-1]$ in $\underline{\mathcal{E}}$. We will write $\Omega_n M$ for the n -fold iteration $\Omega \cdots \Omega M$. Note that $\Omega_n M$ is not in general a well-defined endofunctor of \mathcal{E} , but that it defines an endofunctor of $\underline{\mathcal{E}}$.

Let \mathcal{E} be an idempotent complete exact category. In this section, we will associate to \mathcal{E} a stable ∞ -category $\mathcal{D}_{\text{sing}}(\mathcal{E})$, the singularity ∞ -category of \mathcal{E} , and show that its homotopy category is naturally equivalent to $\underline{\mathcal{E}}$ when \mathcal{E} is Frobenius.

A special case of such a construction can be extracted from Hovey [Hov99, Section 2.2]. A right noetherian ring R is **quasi-Frobenius** if R is injective as a right R -module. See [CR62, Section 58]. In this case, the category Mod_R^\heartsuit of right R -modules is Frobenius, and Hovey constructs a model category structure on Mod_R^\heartsuit whose homotopy category is equivalent to the stable category of Mod_R^\heartsuit . Hovey's construction does not seem to generalize because a small Frobenius category need not embed into a Grothendieck abelian category which is also Frobenius. Specifically, there are examples where the Gabriel-Quillen embedding $\mathcal{E} \rightarrow \mathcal{A}$ does not preserve injectives. Hence, we take a different approach.

Example A.3. Let k be a field and let G be a locally finite group that is not finite (such as \mathbb{Q}/\mathbb{Z}). Then, $k[G]$ is not (right) self-injective by Renault [Ren71], so in particular $\text{Mod}_{k[G]}^\heartsuit$ is not Frobenius. On the other hand, G is the filtered colimit of its finite subgroups, and hence $k[G]$ is the filtered colimit of Frobenius subalgebras along flat transition maps. In particular, $k[G]$ is coherent, for example by [Gla89, Theorem 2.3.3]. It follows that the category of finitely presented (right) $k[G]$ -modules is abelian. It is not hard to check that $k[G]$ is injective in $\text{Mod}_{k[G]}^{\heartsuit, \omega}$, which shows that the category of finitely presented $k[G]$ -modules is Frobenius.

Let \mathcal{E} be an exact category. Let $\text{Ch}^-(\mathcal{E})$ denote the category of bounded below chain complexes of objects in \mathcal{E} . This is a dg category and the dg nerve $N_{\text{dg}}(\text{Ch}^-(\mathcal{E}))$ of [Lur12, Construction 1.3.1.6] is a stable ∞ -category by [Lur12, Proposition 1.3.2.10]. The homotopy category of $N_{\text{dg}}(\text{Ch}^-(\mathcal{E}))$ is the category of bounded chain complexes up to chain homotopy. For simplicity, we will write $\text{Ch}_{\text{dg}}^-(\mathcal{E})$ for $N_{\text{dg}}(\text{Ch}^-(\mathcal{E}))$.

Definition A.4. A complex X in $\text{Ch}^-(\mathcal{E})$ is **acyclic in degree n** if there is a factorization of the differential $X_n \leftarrow X_{n+1} : d_{n+1}$ into

$$X_n \hookleftarrow Z_n \hookleftarrow X_{n+1}$$

where $X_n \hookleftarrow Z_n$ is an admissible mono and a kernel for d_n and $X_{n+1} \twoheadrightarrow Z_n$ is an admissible epi and a cokernel for d_{n+2} . The complex X is **acyclic** if it is acyclic in degree n for all $n \in \mathbb{Z}$.

Consider the following full stable subcategories of $\text{Ch}_{\text{dg}}^-(\mathcal{E})$:

- (i) $\text{Ch}_{\text{dg}}^b(\mathcal{E})$, the dg nerve of the dg category of bounded complexes in \mathcal{E} ;
- (ii) $\text{Ac}_{\text{dg}}^-(\mathcal{E})$ and $\text{Ac}_{\text{dg}}^b(\mathcal{E})$, the dg nerve of the dg category of acyclic bounded below and bounded complexes, respectively, in \mathcal{E} ;
- (iii) $\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E})$ the dg nerve of the dg category of bounded below complexes in \mathcal{E} which are acyclic in all sufficiently high degrees.

Remark A.5. If $\mathcal{E} \subseteq \mathcal{F}$ is fully faithful, then $\text{Ch}^-(\mathcal{E}) \rightarrow \text{Ch}^-(\mathcal{F})$ is fully faithful, which leads to fully faithful maps between all of the subcategories above.

Lemma A.6. *Let \mathcal{E} be an idempotent complete exact category.*

- (a) *The stable ∞ -categories $\text{Ac}_{\text{dg}}^b(\mathcal{E})$ and $\text{Ac}_{\text{dg}}^-(\mathcal{E})$ are idempotent complete in $\text{Ch}_{\text{dg}}^b(\mathcal{E})$ and $\text{Ch}_{\text{dg}}^-(\mathcal{E})$, respectively.*
- (b) *Any chain complex in $\text{Ch}_{\text{dg}}^-(\mathcal{E})$ chain homotopy equivalent to an acyclic chain complex is itself acyclic. In other words, $\text{Ac}_{\text{dg}}^b(\mathcal{E})$ and $\text{Ac}_{\text{dg}}^-(\mathcal{E})$ are closed under equivalence in $\text{Ch}_{\text{dg}}^b(\mathcal{E})$ and $\text{Ch}_{\text{dg}}^-(\mathcal{E})$, respectively.*

Proof. Let $\mathcal{E} \rightarrow \mathcal{A}$ denote the Gabriel-Quillen embedding. We prove first that if $X \in \text{Ch}_{\text{dg}}^-(\mathcal{E})$, then X is acyclic if and only if $H_*(X) = 0$ when X is viewed as a complex of objects in \mathcal{A} . If X is acyclic, then $H_*(X) = 0$ by definition. We can suppose that X is of the form $0 \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots$. Since $H_*(X) = 0$, it follows that $X_0 \leftarrow X_1$ is surjective. Since \mathcal{E} is idempotent complete, it is closed under kernels of admissible epimorphisms in \mathcal{A} . Hence, there is a factorization $X_1 \hookrightarrow Z_1 \hookleftarrow X_2$ where Z_1 is the kernel of $X_0 \leftarrow X_1$ and the cokernel of $X_1 \leftarrow X_2$. By induction, the claim follows.

Now, part (a) follows immediately. Indeed, if $X \simeq Y \oplus Z$ in $\text{Ch}_{\text{dg}}^b(\mathcal{E})$ (resp. $\text{Ch}_{\text{dg}}^-(\mathcal{E})$) where X is acyclic, then $H_*(X) = 0$, so $H_*(Y) = H_*(Z) = 0$. So, Y and Z are bounded (resp. bounded below) complexes with vanishing homology. By the previous paragraph, they are acyclic.

Part (b) follows as well, since if $X \rightarrow Y$ is an equivalence in $\text{Ch}_{\text{dg}}^-(\mathcal{E})$ with X acyclic, we find that $H_*(Y) = 0$, so Y is acyclic. \square

Definition A.7. Let \mathcal{E} be an idempotent complete exact category. The **bounded derived ∞ -category** $\mathcal{D}_{\text{dg}}^b(\mathcal{E})$ of \mathcal{E} is the Verdier quotient

$$\text{Ch}_{\text{dg}}^b(\mathcal{E})/\text{Ac}_{\text{dg}}^b(\mathcal{E}).$$

The homotopy category of $\mathcal{D}_{\text{dg}}^b(\mathcal{E})$ is equivalent to the usual bounded derived category of \mathcal{E} . Similarly, the **bounded below derived ∞ -category** $\mathcal{D}_{\text{dg}}^-(\mathcal{E})$ is the Verdier quotient

$$\text{Ch}_{\text{dg}}^-(\mathcal{E})/\text{Ac}_{\text{dg}}^-(\mathcal{E}),$$

while the **homologically bounded derived ∞ -category** $\mathcal{D}_{\text{dg}}^{-,b}(\mathcal{E})$ is the Verdier quotient

$$\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E})/\text{Ac}_{\text{dg}}^{-,b}(\mathcal{E}).$$

A map in $\text{Ch}_{\text{dg}}^-(\mathcal{E})$ which is an equivalence in any of these derived ∞ -categories is called a **quasi-isomorphism**. By Lemma A.6(b), these are precisely the maps in $\text{Ch}_{\text{dg}}^-(\mathcal{E})$ whose cones are acyclic.

Remark A.8. If \mathcal{A} is a small abelian category viewed as an exact category in the usual way, then $\mathcal{D}_{\text{dg}}^b(\mathcal{A}) \simeq \mathcal{D}^b(\mathcal{A})$, where $\mathcal{D}^b(\mathcal{A})$ is defined as in Section 3.3 as $\tilde{\mathcal{D}}(\text{Ind}(\mathcal{A}))^\omega$.

Proposition A.9. *The natural functor $\mathcal{D}_{\text{dg}}^b(\mathcal{E}) \rightarrow \mathcal{D}_{\text{dg}}^{-,b}(\mathcal{E})$ is an equivalence and the natural functor $\mathcal{D}_{\text{dg}}^{-,b}(\mathcal{E}) \rightarrow \mathcal{D}_{\text{dg}}^-(\mathcal{E})$ is fully faithful.*

Proof. The second functor is trivially fully faithful since $\text{Ac}_{\text{dg}}^-(\mathcal{E}) \subseteq \text{Ch}_{\text{dg}}^{-,b}(\mathcal{E})$. We prove that the composition is fully faithful. For this, it suffices to verify Verdier's criterion [Ver96, Proposition II.2.3.5]. Thus, if $f : M \rightarrow X$ is a map in $\text{Ch}_{\text{dg}}^-(\mathcal{E})$ with M in $\text{Ac}_{\text{dg}}^-(\mathcal{E})$ and X in $\text{Ch}_{\text{dg}}^b(\mathcal{E})$, we show that f factors through $M \rightarrow M'$ where M' is in $\text{Ac}_{\text{dg}}^b(\mathcal{E})$. Choose n such that $X_i = 0$ for $i \geq n$. Since M is acyclic, the good truncation $\tau_{\leq n} M$ exists in $\text{Ch}_{\text{dg}}^b(\mathcal{E})$ and is also acyclic. The map $M \rightarrow X$ factors through $M \rightarrow \tau_{\leq n} M$ since $X_n = 0$.

To see essential surjectivity, let X be in $\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E})$ and choose n such that X is acyclic in degrees n and higher. Then, the good truncation $\tau_{\leq n} X$ exists in $\text{Ch}_{\text{dg}}^b(\mathcal{E})$ and $X \rightarrow \tau_{\leq n} X$ is a quasi-isomorphism because the cone has zero homology and is hence acyclic by the argument in the proof of Lemma A.6. \square

Theorem A.10 (Balmer-Schlichting [BS01, Theorem 2.8]). *If \mathcal{E} is idempotent complete, then the derived ∞ -category $\mathcal{D}_{\text{dg}}^b(\mathcal{E})$ is idempotent complete.*

Proof. This can be checked on the homotopy category, which is done in [BS01]. \square

Lemma A.11. *Any complex P in $\text{Ac}_{\text{dg}}^-(\mathcal{E}) \cap \text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}})$ is contractible.*

Proof. This follows immediately from the projectivity of the terms of P . \square

Remark A.12. It follows that $\text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}}) \simeq \mathcal{D}_{\text{dg}}^b(\mathcal{E}^{\text{proj}})$ and $\text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}}) \simeq \mathcal{D}_{\text{dg}}^-(\mathcal{E}^{\text{proj}})$, since the acyclic complexes are already equivalent to zero in $\text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}})$.

Corollary A.13. *The stable ∞ -category $\text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}})$ is idempotent complete.*

Proof. This is a special case of Theorem A.10. \square

Lemma A.14. *If X is in $\text{Ac}_{\text{dg}}^-(\mathcal{E})$ and P is in $\text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}})$, then any map $f : P \rightarrow X$ is chain homotopic to zero.*

Proof. We assume that $P_n = 0$ for $n \leq -1$. Let $s_n : P_n \rightarrow X_{n+1}$ be the zero map for $n \leq -1$. Assume that s_n has been constructed for $n \leq N-1$ such that $f_i = d_{i+1}^X \circ s_i + s_{i-1} \circ d_i^P$ for $i \leq N-1$. Then,

$$\begin{aligned} d_N^X \circ (f_N - s_{N-1} \circ d_N^P) &= d_{N-1}^X \circ f_N - d_N^X \circ s_{N-1} \circ d_N^P \\ &= d_{N-1}^X \circ f_N - (f_{N-1} - s_{N-2} \circ d_{N-1}^P) \circ d_N^P \\ &= d_{N-1}^X \circ f_N - f_{N-1} \circ d_N^P \\ &= 0 \end{aligned}$$

since f is a map of chain complexes. It follows from the acyclicity of X that $f_N - s_{N-1} \circ d_N^P$ factors through the admissible mono $Z_N \hookrightarrow X_N$. Since $X_{N+1} \rightarrow Z_N$ is an admissible epi, there is a lift of $f_N - s_{N-1} \circ d_N^P$ to $s_N : P_N \rightarrow X_{N+1}$ such that $d_{N+1}^X \circ s_N = f_N - s_{N-1} \circ d_N^P$. By induction, this proves the existence of a contracting homotopy for f . \square

Proposition A.15. *The functors $\text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}}) \rightarrow \mathcal{D}_{\text{dg}}^b(\mathcal{E})$ and $\text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}}) \rightarrow \mathcal{D}_{\text{dg}}^-(\mathcal{E})$ are fully faithful.*

Proof. We use Verdier's criterion [Ver96, Proposition II.2.3.5], which says in our case that if every map $P \rightarrow X$ with P in $\text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}})$ and X in $\text{Ac}_{\text{dg}}^-(\mathcal{E})$ factors through a map $X' \rightarrow X$ where X' is in $\text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}}) \cap \text{Ac}_{\text{dg}}^-(\mathcal{E})$, then

$$\text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}}) / \text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}}) \cap \text{Ac}_{\text{dg}}^-(\mathcal{E}) \rightarrow \mathcal{D}_{\text{dg}}^-(\mathcal{E})$$

is fully faithful. But, Lemma A.14 says that in fact every such map factors through zero, so the criterion is satisfied. On the other hand, Lemma A.11 says every complex in $\text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}}) \cap \text{Ac}_{\text{dg}}^-(\mathcal{E})$ is already equivalent to zero, so that the conclusion of Verdier's criterion reduces to the statement of the proposition. The bounded case is similar, or follows from the fully faithfulness of $\text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}}) \rightarrow \text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}})$ and $\mathcal{D}_{\text{dg}}^b(\mathcal{E}) \rightarrow \mathcal{D}_{\text{dg}}^-(\mathcal{E})$. \square

Corollary A.16. *The natural functor $\text{Ch}_{\text{dg}}^-(\mathcal{E}^{\text{proj}}) \rightarrow \mathcal{D}_{\text{dg}}^-(\mathcal{E})$ is an equivalence.*

Proof. Thanks to the previous proposition it suffices to check essential surjectivity, which follows by taking projective resolutions. \square

Definition A.17. Let \mathcal{E} be an idempotent complete exact category. The **singularity ∞ -category** $\mathcal{D}_{\text{sing}}(\mathcal{E})$ of \mathcal{E} is the Verdier quotient

$$\mathcal{D}_{\text{dg}}^b(\mathcal{E}) / \text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}}).$$

We will write $\mathcal{D}_{\text{sing}}$ for the induced functor from the ∞ -category of exact categories and exact functors to $\text{Cat}_{\infty}^{\text{perf}}$.

Definition A.18. Syzygys play a crucial role in the proof of the next theorem. Let X in $\text{Ch}_{\text{dg}}^-(\mathcal{E})$ be acyclic in degree $n - 1$. Then, the n th syzygy $\Omega_n X$ is an object of \mathcal{E} , being the kernel of $d_{n-1} : X_{n-1} \rightarrow X_{n-2}$. Moreover, in this case, the brutal truncation $\sigma_{\geq n} X$ admits a canonical map to $\Omega_n X[n]$. When X is acyclic, $\sigma_{\geq n} X \rightarrow \Omega_n X[n]$ is a quasi-isomorphism. Finally, if X is a complex of projectives which is acyclic in degree i for $i \geq n - 1$, then $\Omega_i X \cong \Omega_{i-n} \Omega_n X$ in $\underline{\mathcal{E}}$.

Theorem A.19. *There is a natural equivalence $\text{Ho}(\mathcal{D}_{\text{sing}}(\mathcal{E})) \simeq \underline{\mathcal{E}}$ when \mathcal{E} is an idempotent complete Frobenius category.*

Proof. The proof of this theorem is due in spirit to Buchweitz [Buc86], though only a special case is given there. For simplicity, we avoid the comparison with the homotopy category of acyclic complexes of projectives, instead giving a direct argument for the equivalence.

There is an evident composition of functors $N(\mathcal{E}) \rightarrow \text{Ch}_{\text{dg}}^b(\mathcal{E}) \rightarrow \mathcal{D}_{\text{dg}}^b(\mathcal{E}) \rightarrow \mathcal{D}_{\text{sing}}(\mathcal{E})$, where the first is given by viewing an object of \mathcal{E} as a chain complex concentrated in degree zero. This first functor is evidently fully faithful. The second and third functors are the Verdier quotient functors.

Let $\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E}^{\text{proj}})$ be the full subcategory of $\text{Ch}^-(\mathcal{E}^{\text{proj}})$ consisting of **homologically bounded complexes of projectives**, i.e., those complexes which are acyclic when viewed in $\text{Ch}^-(\mathcal{E})$ in all sufficiently high degrees. It is clear that the natural functor $\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E}^{\text{proj}}) \rightarrow \mathcal{D}^-(\mathcal{E})$ induces an equivalence $\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E}^{\text{proj}}) \simeq \mathcal{D}^{-,b}(\mathcal{E})$. Hence, there are equivalences

$$\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E}^{\text{proj}})/\text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}}) \simeq \mathcal{D}_{\text{dg}}^{-,b}(\mathcal{E})/\text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}}) \simeq \mathcal{D}_{\text{sing}}(\mathcal{E}).$$

We are therefore free in our arguments to replace bounded complexes in \mathcal{E} with homologically bounded complexes of projectives.

We claim first that the functor $\underline{\mathcal{E}} \rightarrow \mathcal{D}_{\text{sing}}(\mathcal{E})$ is essentially surjective. Pick a complex X of $\text{Ch}_{\text{dg}}^b(\mathcal{E})$, and let $P \rightarrow X$ be a quasi-isomorphism where P is a bounded below complex of projectives. Choose $n \geq 0$ sufficiently large so that P is acyclic in degree i for all $i \geq n$. In this case, the brutal truncation $\sigma_{\geq i} P$ admits a quasi-isomorphism $\sigma_{\geq i} P \rightarrow \Omega_i P[i]$ for all $i > n$, where $\Omega_i P$ is some object of \mathcal{E} . Fix $i > n$ and extend $\sigma_{\geq i} P$ to an unbounded acyclic complex Q of projectives, by taking a projective co-resolution of $\Omega_i P$ (which exists because \mathcal{E} is Frobenius). There is a diagram of morphisms

$$X \leftarrow P \rightarrow \sigma_{\geq i} P = \sigma_{\geq i} Q \leftarrow \sigma_{\geq 0} Q \rightarrow \Omega_0 Q$$

in $\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E})$. The outside arrows are quasi-isomorphisms and hence already equivalences in $\mathcal{D}_{\text{dg}}^{-,b}(\mathcal{E})$. The inside arrows have cones in $\text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}})$, and hence they become equivalences in $\mathcal{D}_{\text{sing}}(\mathcal{E})$. But, this shows that $X \simeq \Omega_0 Q$ in $\mathcal{D}_{\text{dg}}^{-,b}(\mathcal{E})/\text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}}) \simeq \mathcal{D}_{\text{sing}}(\mathcal{E})$.

To finish the proof, it is enough to prove that $\underline{\mathcal{E}} \rightarrow \text{Ho}(\mathcal{D}_{\text{sing}}(\mathcal{E}))$ is fully faithful. We check faithfulness and fullness separately. Let M and N be objects of \mathcal{E} . Since $\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E}^{\text{proj}}) \rightarrow \mathcal{D}_{\text{dg}}^b(\mathcal{E})$ is fully faithful, by replacing M and N by projective resolutions, we see that $\text{Hom}_{\mathcal{E}}(M, N) \rightarrow \pi_0 \text{Map}_{\mathcal{D}_{\text{dg}}^b(\mathcal{E})}(M, N)$ is a bijection. Now, consider a diagram

$$M \xleftarrow{s} X \xrightarrow{f} N$$

in $\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E}^{\text{proj}})$ with $\text{cone}(s) \in \text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}})$. Then, $\Omega_n M \leftarrow \Omega_n X$ is an isomorphism up to projective summands for n sufficiently large, so there is an induced map $\Omega_n M \rightarrow \Omega_n N$. Since Ω_n is an autoequivalence of $\underline{\mathcal{E}}$, fullness follows.

To prove faithfulness, suppose that $f : M \rightarrow N$ maps to zero in $\pi_0 \text{Map}_{\mathcal{D}_{\text{sing}}(\mathcal{E})}(M, N)$. Then, there is $X \xrightarrow{s} M$ such that $\text{cone}(s)$ is quasi-isomorphic to an object of $\text{Ch}_{\text{dg}}^b(\mathcal{E}^{\text{proj}})$ and $f \circ s$ is zero in $\mathcal{D}_{\text{dg}}^b(\mathcal{E})$. Working with bounded below complexes of projectives, we can assume in fact that $f \circ s$ is nullhomotopic in $\text{Ch}_{\text{dg}}^{-,b}(\mathcal{E}^{\text{proj}})$. In this case, $M \rightarrow N$ factors through $X \rightarrow \text{cone}(s)$. A sufficiently high syzygy of $\text{cone}(s)$ is

projective, so this means that $\Omega_n f$ factors through a projective, and hence is zero in $\underline{\mathcal{E}}$. Again using that Ω_n is an autoequivalence, we find that $f = 0$ in $\underline{\mathcal{E}}$, as desired. \square

Example A.20. In general $\mathcal{D}_{\text{sing}}(\mathcal{E})$ is not idempotent complete, and hence neither is $\underline{\mathcal{E}}$. It is enough to find a Gorenstein noetherian commutative ring R with $K_{-1}(R) \neq 0$, since in this case there is an isomorphism

$$K_0(\tilde{\mathcal{E}})/K_0(\underline{\mathcal{E}}) \cong K_{-1}(R)$$

(as $K_{-1}(\mathcal{D}_{\text{dg}}^b(R)) = 0$ by Schlichting’s theorem). The complete intersection $R = \mathbb{Z}[x_0, x_1]/(x_0 x_1(1 - x_0 - x_1))$ works. In this case, $K_{-1}(R) \cong \mathbb{Z}$, as can be checked from [Wei84].

Let R be a noetherian commutative ring. The abelian category $\text{Mod}_R^{\heartsuit, \omega}$ of finitely presented discrete R -modules is exact, and its negative K -theory vanishes by Schlichting. Hence, $K_{-n}(\mathcal{D}_{\text{sing}}(\text{Mod}_R^{\heartsuit, \omega})) \rightarrow K_{-n-1}(R)$ is a surjection for $n \geq 0$ and an isomorphism for $n \geq 1$. In this way, the singularity category supports the negative K -theory of R and gives one measurement of the singularities of R itself.

When R is not noetherian, the question of whether or not this connection continues is precisely bound up in Schlichting’s conjecture. For example, if R is merely coherent, then it is no longer known in general that $K_{-n}(\mathcal{D}_{\text{sing}}(\text{Mod}_R^{\heartsuit, \omega})) \rightarrow K_{-n-1}(R)$ is an isomorphism for $n \geq 1$. This would follow from Conjecture A.

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